SOLUTIONS TO QUIZ #8

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1. Problem 1

We use the formula

$$S = \int_R dS = \int_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}dA = \int_R \sqrt{4x^2+4y^2+1}dA.$$

This can be most simply expressed in cylindrical coordinates:

$$\int_R \sqrt{4x^2+4y^2+1}dA = \int_0^{\pi/2}\int_0^{\sqrt{5}} \left(\sqrt{4r^2+1}\right) r\,dr\,r\,d\theta,$$

which transforms under the substitutions $2r = \tan u$ and $t = \sec u$ to

$$\int_0^{\pi/2}\int_0^{\sqrt{5}} \left(\sqrt{4r^2+1}\right) r\,dr\,r\,d\theta = \int_0^{\pi/2}\int_{r=0}^{\sqrt{5}} \frac{1}{4}\tan u \sec^3 u\,du\,d\theta = \frac{\pi}{2}\cdot\frac{1}{4}\int_{r=0}^{r=\sqrt{5}} t^2 dt.$$

These substitutions pair to give $t = \sec u = \sqrt{4r^2+1}$, and substituting that into the integral gives

$$\frac{\pi}{2}\cdot\frac{1}{4}\left(\frac{t^3}{3}\right)\Bigg|_{r=0}^{r=\sqrt{5}} = \frac{\pi}{2}\cdot\left(\frac{1}{4}\cdot\frac{1}{3}\left(\sqrt{(2r)^2+1}\right)^3\Bigg|_{r=0}^{\sqrt{5}}\right) = \frac{\pi}{24}\cdot\left(21\sqrt{21}-1\right).$$

2. Problem 2

We make an immediate switch to cylindrical coordinates.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2+y^2)^{5/2} dy\,dx = \int_0^2\int_0^{2\pi} r^5\cdot r\,d\theta\,dr.$$

Now this is not so bad:

$$\int_0^2\int_0^{2\pi} r^5\cdot r\,d\theta\,dr = 2\pi\cdot\frac{r^7}{7}\Bigg|_0^2 = \frac{256}{7}\pi.$$

3. Problem 3

I like to just draw the picture, but a more systematic thing to do is to chain together the inequalities dictated by the bounds: $0 \le z \le 1$, $0 \le y \le z^2$, and $0 \le x \le y$ collectively become $0 \le x \le y \le z^2 \le 1$. From here, to read off the appropriate bounds, we notice only that iterated integrals have bounds which are successively dependent upon each other. For instance, the first integral uses "$dy\,dz\,dx$", so y is bounded by x and z^2, then z is bounded by $\sqrt{x}$ and 1 (since it is not allowed to depend upon y, and x is bounded by 0 and 1 (since it is not allowed to depend upon y or z.[1] Following similar reasoning for the other integral, we get the formulas

$$\int_0^1\int_0^{z^2}\int_0^y f\,dx\,dy\,dz = \int_0^1\int_{\sqrt{x}}^1\int_x^{z^2} f\,dy\,dz\,dx = \int_0^1\int_0^y\int_{\sqrt{y}}^1 f\,dz\,dx\,dy.$$

[1] It's more precise to say that we're picking these rewritten inequalities because, collectively, they generate the same total inequality. This is actually sort of important; hidden in this reasoning is the extra inequality $0 \le z \le z^2 \le 1$.

4. Problem 4

Set up the integral:

$$V = \int_0^1 \int_z^1 \int_y^1 dx\,dy\,dz,$$

and compute:

$$\int_0^1 \int_z^1 \int_y^1 dx\,dy\,dz = \int_0^1 \int_z^1 (1-y)\,dy\,dz = \int_0^1 \left(\frac{1}{2} - z + \frac{z^2}{2}\right) dz = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}.$$