SOLUTIONS TO QUIZ #7

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1. Problem 1

Like last week, in the presence of a constraint equation we're to apply Lagrange multipliers. Take f as in the problem and set $C(x,y,z) = x^2 + y^2 + z^2$ to be the constraint equation; then we're to solve

$$\nabla f = \lambda \cdot \nabla C.$$

Calculating partial derivatives gives

$$\begin{pmatrix} 4x^3 \\ 4y^3 \\ 4z^3 \end{pmatrix} = \lambda \cdot \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}.$$

Supposing that none of x, y, or z are zero, we can divide the first, second, and third equations by these respective values to get

$$\begin{pmatrix} 4x^2 \\ 4y^2 \\ 4z^2 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ 2\lambda \\ 2\lambda \end{pmatrix}.$$

Summing these equations and applying the constraint gives

$$4(x^2 + y^2 + z^2) = 4 = 6\lambda,$$

and hence $\lambda = 2/3$. We are now in the position to solve each equation: $4x^2 = 4/3$, for instance, gives $x = \pm\sqrt{1/3}$, and similarly for y and z. At each of these eight points, f takes the value $3 \cdot (\sqrt{1/3})^4 = 1/3$.

Now suppose instead that just one of the coordinates is zero. Because the function is symmetric, for our analysis we may as well assume that coordinate to be x, while y and z are nonzero. In this case, the system of equations reduces to

$$\begin{pmatrix} 0 \\ 4y^3 \\ 4z^3 \end{pmatrix} = \lambda \cdot \begin{pmatrix} 0 \\ 2y \\ 2z \end{pmatrix}$$

with constraint $y^2 + z^2 = 1$. We now only need to add the bottom two equations to apply the method from the previous paragraph and conclude $\lambda = 1$. Solving $4y^2 = 2$ gives $y = \pm\sqrt{1/2}$ and similarly for z. At these points, f takes the value $2 \cdot (\sqrt{1/2})^4 = 1/2$. Finally, if two coordinates are taken to be zero, the constraint forces that the final coordinate take the value ± 1, and f takes the value 1 there. It follows that the minimum value of f is 1/3 and the maximum value is 1.

2. Problem 2

The function describing the distance from a point (x,y,z) to the origin is $d(x,y,z) = \sqrt{x^2 + y^2 + z^2}$.[1] The problem instructs us to consider the constraint function $C(x,y,z) = x^3 + y^3 + z^3$ and the level curve $C(x,y,z) = 1$.

Following Lagrange's method, we calculate

$$\begin{pmatrix} x/d \\ y/d \\ z/d \end{pmatrix} = \lambda \cdot \begin{pmatrix} 3x^2 \\ 3y^2 \\ 3z^2 \end{pmatrix},$$

[1] You can solve this more simply by using $f = d^2$, which has the same extremal behavior as d, but this is not strictly necessary.

or

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = d\lambda \cdot \begin{pmatrix} 3x^2 \\ 3y^2 \\ 3z^2 \end{pmatrix}.$$

The first equation is only solvable if $x = 0$ or if $x \neq 0$ and $d\lambda = 1/(3x)$, and similarly for the other three equations. Again, because both f and the constraint are symmetric under permuting x, y, and z, we can break into the cases where none of the points are zero, where $x = 0$ only, and where $x = 0$ and $y = 0$. (It's not possible for all three to be zero, since then the constraint is not satisfiable.)

If none of the points are zero, then they are all equal to $1/(3d\lambda)$ and hence to each other. Substituting this into the constraint, we find $x = y = z = 1/3^{1/3}$, which has distance

$$d(1/3^{1/3}, 1/3^{1/3}, 1/3^{1/3}) = \sqrt{3 \cdot 1/3^{2/3}} = 3^{1/6}.$$

If $x = 0$ alone, then y and z are again both equal to $1/(3d\lambda)$ and hence to each other. Substituting this into the constraint, we find $y = z = 1/2^{1/3}$, which has distance

$$d(0, 1/2^{1/3}, 1/2^{1/3}) = \sqrt{2 \cdot 1/2^{2/3}} = 2^{1/6}.$$

Similarly, there are the other two points $(1/2^{1/3}, 0, 1/2^{1/3})$ and $(1/2^{1/3}, 1/2^{1/3}, 0)$. Finally, if $x = 0$ and $y = 0$ both, then $z = 1$ which has distance

$$d(0,0,1) = 1.$$

Similarly, there are the other two points $(0,1,0)$ and $(1,0,0)$.

Since this last group of points has the smallest distance values, they minimize the distance function globally on $C = 1$.

3. Problem 3

The volume integral is set up as

$$\int_0^1 \left(\int_x^1 \left(e^{x-y} - (-e^{x-y})\right) dy \right) dx.$$

Now we just compute:

$$\begin{aligned} \int_0^1 \int_x^1 2e^{x-y}\, dy\, dx &= \int_0^1 \left(-2e^{x-y}\Big|_{y=x}^1 \right) dx = \int_0^1 (2 - 2e^{x-1}) dx \\ &= 2x - 2e^{x-1}\Big|_{x=0}^1 = 2 - 2 - 0 + 2e^{-1} = 2e^{-1}. \end{aligned}$$

4. Problem 4

The region of integration is constrained by the curves $y = 0$, $x = y$, and $x = \sqrt{\pi}$. To set up the bounds for the other order of integration, we consider first the possible x–values: they must lie between 0 and $\sqrt{\pi}$. Then, given an x–value, the possible y–values for that x–value lie between 0 and x. So, we have

$$\begin{aligned} \int_0^{\sqrt{\pi}} \left(\int_x^{\sqrt{\pi}} \cos(x^2) dy \right) dx &= \int_0^{\sqrt{\pi}} \left(y\cos(x^2)\Big|_{y=0}^x \right) dx = \int_0^{\sqrt{\pi}} \left(x\cos(x^2) \right) dx \\ &= \frac{1}{2}\sin(x^2)\Big|_{x=0}^{\sqrt{\pi}} = \frac{1}{2}(0 - 0) = 0. \end{aligned}$$