SOLUTIONS TO QUIZ #6

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1. Problem 1

A point (x_0, y_0) is a critical point of f when $\nabla f|_{(x_0,y_0)}$ is the zero–vector (or does not exist).[1] So, to start, we should calculate ∇f:

$$\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} = \begin{pmatrix} -x^2+1 \\ -2y \end{pmatrix}.$$

The points x_0 with $-x_0^2+1=0$ are $x_0=\pm 1$, and the points y_0 with $-2y_0=0$ are $y_0=0$.

To describe these critical points, we apply the second derivative test, which requires us to calculate the quantity[2]

$$Df = \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

In our case, this function is given by

$$Df = (-2x)(-2) - 0 = 4x,$$

which at the point $(1,0)$ gives the positive value 4 (and $\partial^2 f/\partial x^2|_{(1,0)} = -2$) and at the point $(-1,0)$ gives the negative value -4. According to the book, the point $(1,0)$ is thus a local maximum and $(-1,0)$ is neither a saddle point.

2. Problem 2

Let l, w, and h denote the length, width, and height of our box respectively. We are to minimize the cost function specified by

$$C(l,w,h) = 2\cdot lw\cdot 8 + 2\cdot wh\cdot 1 + 2\cdot lh\cdot 1,$$

subject to the constraint that the volume $V(l,w,h) = lwh$ satisfies $V(l,w,h)=8$. The method of Lagrange multipliers instructs us to solve the equation

$$\nabla C = \lambda\cdot\nabla V$$

for some scalar λ. Inserting our definitions of C and V, we produce the system of three equations

$$\begin{pmatrix} 16w+2h \\ 16l+2h \\ 2w+2l \end{pmatrix} = \lambda\cdot\begin{pmatrix} wh \\ lh \\ lw \end{pmatrix}.$$

First multiplying the first equation by l, the second equation by w, and the third equation by h and then applying the constraint $V(l,w,h) = lwh = 8$, we produce the new system

$$\begin{pmatrix} 16lw+2lh \\ 16lw+2wh \\ 2wh+2lh \end{pmatrix} = \begin{pmatrix} 8\lambda \\ 8\lambda \\ 8\lambda \end{pmatrix}.$$

Ignoring λ, which is an auxiliary variable, we now see that all three quantities in the left–hand vector are equal to each other. Equating the first two, we find $lh = wh$, which in terms of our original system gives $16l+2h = 16w+2h$, and hence $w=l$. The last equation of the original system then gives $4l = \lambda l^2$ and hence $\lambda l = 4$ (as the other solution $l=0$ cannot satisfy our constraint equation). Applying this to the middle equation gives $h = 8l$, from which follows

$$(l,w,h) = (1,1,8).$$

[1] This is a special case of a more general condition, called "the Jacobian of f drops rank".

[2] This is sometimes called the "Hessian".

3. Problem 3

To solve an optimization problem on a domain with boundary, we have to solve two subproblems: we can optimize the function on the interior of the domain using the methods of Problem 1, and we can optimize the function on the boundary using the methods of Problem 2. (The quiz hints that Lagrange multipliers are not necessary here, but I'm going to use them anyway. No harm in being systematic.)

For the first part, we compute

$$\nabla f = \begin{pmatrix} 12x^2 \\ 6y \end{pmatrix}.$$

The equation $\nabla f = 0$ is satisfied by $(x, y) = (0, 0)$ alone. This point is in the interior of the disk, so it is relevant to us, and we should classify what sort of critical point it is. If we were to calculate the Hessian, we would find that it is zero, and so we cannot apply the Second Derivative Test. However, it is easy enough to check this function by hand: along the x–axis (i.e., when $y = 0$), it looks like the function $f(x, 0) = 4x^3$, which has neither a minimum nor a maximum at $x = 0$, and so our function is not extremized at $(0, 0)$ either.

For the second part, we will solve the equation

$$\nabla f = \lambda \cdot \nabla C,$$

where $C(x, y) = x^2 + y^2$ describes the constraint level curve: $C(x, y) = 1$. We compute:

$$\begin{pmatrix} 12x^2 \\ 6y \end{pmatrix} = \lambda \cdot \begin{pmatrix} 2x \\ 2y \end{pmatrix}.$$

The second equation forces $\lambda = 3$ or $y = 0$, and we consider these cases separately. In the case that $\lambda = 3$, the equation $12x^2 = 6x$ has solutions $x = 0$ and $x = 1/2$. We are thus left (together with the case $y = 0$ above) with the possible points $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$, $(1/2, \sqrt{3}/2)$, and $(1/2, -\sqrt{3}/2)$ to check by hand. At these points, f takes on the values

$$f(1, 0) = 4, \qquad f(-1, 0) = -4, \qquad f(0, 1) = f(0, -1) = 3, \qquad f\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = f\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = 11/4.$$

It follows immediately that $(1, 0)$ maximizes the function and $(-1, 0)$ minimizes it.

4. Problem 3, redux

We can also, as indicated in the problem text, solve the boundary part of the optimization problem without Lagrange multipliers. To do this, recall that we have a parametrization of the unit circle given by $\gamma(t) = (\cos t, \sin t)$. We are interested in the differential behavior of f, and the multivariate chain rule tells us that the derivative of $f \circ \gamma$ comes as the dot product of the gradient of f with the component-wise derivative of γ. Since γ' is *never* zero, any zeroes of the derivative of this composite function must correspond to critical points of f.

So, we compute: $(f \circ \gamma)(t) = 4\cos^3(t) + 3\sin^2(t)$ has derivative

$$(f \circ \gamma)' = 6 \sin t \cos t - 12 \cos^2 t \sin t = 6 \sin t \cos t (1 - 2 \cos t).$$

So, if $(f \circ \gamma)'(t) = 0$, then either $\sin t = 0$, $\cos t = 0$, or $\cos t = 1/2$. These occur at the t–values

$$t \in \{0, \pi/3, \pi/2, 2\pi/3, \pi, 3\pi/2\}.$$