Practice Math 25b Final #1

This is a practice midterm for Math 25b. To give you the experience of a real exam, we include the following bits of information and warnings.

- *Do not* open the test booklet until told to do so.
- There are *10* questions on this midterm. Make sure you have all of them.
- There are two pages for scratch work included at the end.
- No outside materials are allowed for reference: no friends, no phones, no books, no notes, no pages from other exams, no wandering eyes, ... The only thing you may use for the duration of this exam is your pencil.
- The exam is to last 50 minutes.
- You are allowed to cite results from Axler, from Spivak, or from the classroom. (The one exception is if a question were to ask you to re-prove such a result. Stating "we did this in class" is not a sufficient answer in that case.)
- Agree to the following by signing on the blank line:

  I, ______________________________, am bound by the Harvard Honor Code, which I recently signed when registering for classes. Accordingly, I understand the serious consequences that would befall me if I were to cheat on this midterm. I hereby affirm that I have not cheated.

Unsigned exams will be left ungraded and the examinee marked as absent.

Problem 1. Suppose $\sum_{i=0}^{\infty} x_i$ is a convergent series in $\mathbb{R}^n$. Show that the triangle inequality applies to the infinite sum:

$$\left\| \sum_{i=0}^{\infty} x_i \right\| \leq \sum_{i=0}^{\infty} \|x_i\|.$$

Problem 2. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *homogeneous of degree* m when it satisfies the equation

$$f(t \cdot x) = t^m \cdot f(x)$$

for any scalar t. Suppose that f is also differentiable, and demonstrate

$$\sum_{j=1}^{n} x_j \cdot \frac{\partial f}{\partial x_j}(x) = m \cdot f(x).$$

Problem 3. Let (a_i) be an enumeration of the rationals in $[0,1]$, i.e., $\{a_i \mid i \geq 0\} = \mathbb{Q} \cap [0,1]$ and each value a_i is distinct. Define a function $f_k\colon\ [0,1] \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} 1 & \text{if } x \in [0,1] \text{ and } |x - a_i| < 10^{-i} \text{ for } i \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Show that each f_k is Riemann integrable but that they cannot each be modified on a set of measure zero to converge to a Riemann integrable function.

Problem 4. Show that a quadratic form $Q(x_1, \ldots, x_n)$ can be represented by a unique symmetric $n \times n$ matrix A satisfying the formula

$$Q(x) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \cdot A \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Problem 5. Let $f\colon [0,1] \to \mathbb{R}$ a continuous function such that $f(0) = f(1) = 0$, and let $n \in \mathbb{N}$ be some fixed value. Prove there is a real value $x \in [0,1]$ such that $f(x) = f(x + \frac{1}{n})$.

Problem 6. Let $M \subseteq \mathbb{R}^n$ be a manifold of the form $M = g^{-1}(0)$ for some C^1–function $g\colon \mathbb{R}^n \to \mathbb{R}$ satisfying $D_x g \neq 0$ for all $x \in M$. Show that M has a consistent orientation.

Problem 7. A function $f\colon \mathbb{R} \to \mathbb{R}$ is a *local diffeomorphism* if for every $x \in \mathbb{R}$, there is an interval $(x-\delta, x+\delta)$, and f restricted to $(x-\delta, x+\delta)$ is a diffeomorphism (i.e., it is C^∞, it is invertible, and its inverse is C^∞). Prove that if f is a local diffeomorphism, then its image is an open interval or $\mathbb{R}$, and it is in fact a diffeomorphism.

Problem 8.

1. Parameterize the surface in 4–space given by the equations

$$x_1^2 + x_2^2 = a^2, \qquad x_3^2 + x_4^2 = b^2.$$

2. Integrate the 2–form $x_1 x_2 \, \mathrm{d}x_2 \wedge \mathrm{d}x_3$ over this surface.
3. Compute the derivative of this 2–form.
4. Present the surface as the boundary of a 3–manifold and verify Stokes's theorem as applied to this integral.

Problem 9. Suppose $U \subseteq \mathbb{R}^3$ is open, ω is a 2–form on U, $x \in U$ is some point, and $S_r(x)$ is the sphere of radius r centered at x. Compute

$$\lim_{r \to 0} \frac{1}{r^3} \int_{S_r(x)} \omega.$$

Problem 10. Let V a vector space of dimension n. Prove that the wedge product map

$$\wedge : \mathrm{Alt}^p(V) \times \mathrm{Alt}^{n-p}(V) \to \mathrm{Alt}^n(V)$$

induces an isomorphism of $\mathrm{Alt}^p(V)$ with the dual of $\mathrm{Alt}^{n-p}(V)$.