LOCAL COORDINATES AND THE IMPLICIT FUNCTION THEOREM

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At the end of Spivak's section on the implicit function theorem, he states the following result:

Corollary 1. *Let* $f\colon \mathbb{R}^n \to \mathbb{R}^r$ *be continuously differentiable in an open set containing* a*, where* $p \leq n$*. If* $D_p f$ *has rank* r*, then there is an open set* $A \subseteq \mathbb{R}^n$ *containing* p*, a second open set* B*, and a differentiable function* $h\colon B \to A$ *with differentiable inverse such that*

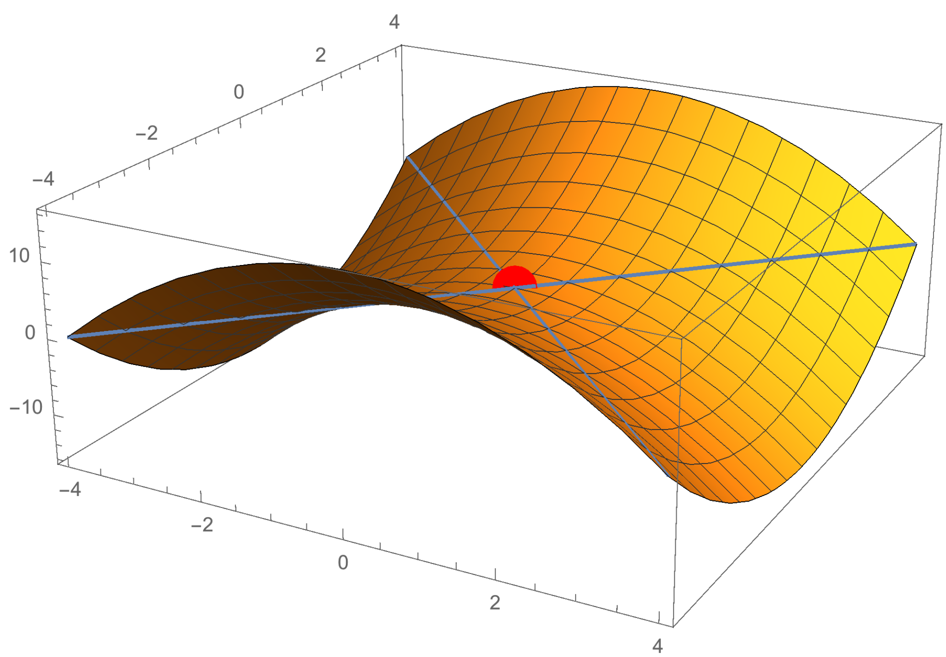
$$f \circ h(x_1, \ldots, x_n) = (x_{n-r+1}, \ldots, x_n). \quad \square$$

He doesn't do an especially good job of telling you *why* you would care about this theorem. The hypothesis that $D_a f$ has rank p exactly means that, up to change of basis (using horizontal and vertical Gaussian elimination), the matrix expressing $D_a f$ can be written in the form

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right).$$

The core idea of the corollary, then, is that h is a kind of Gaussian elimination *for* f *itself*, so that not only does $D_{h^{-1}(a)}(f \circ h)$ take the form of the centered matrix, but $f \circ h$ itself (i.e., without linearization) literally looks like that centered matrix too.

This is easier to see in an example. Take $f(x, y) = y^2 - x^2$, a function whose graph familiar from the practice midterm:



You've already calculated its derivative to be $D_{(p_x,p_y)} f = \begin{pmatrix} -2p_x & 2p_y \end{pmatrix}$. So, if we pick a point like $p = (1, 0)$, we get $D_{(1,0)} = \begin{pmatrix} -2 & 0 \end{pmatrix}$, which is of full rank. The theorem thus applies, guaranteeing the existence of a locally defined function h, comparing our standard coordinates (x, y) with some other coordinates (a, b), and which have the property $f \circ h(a, b) = a$. Thinking of (a, b) as functions of (x, y), this gives the equation

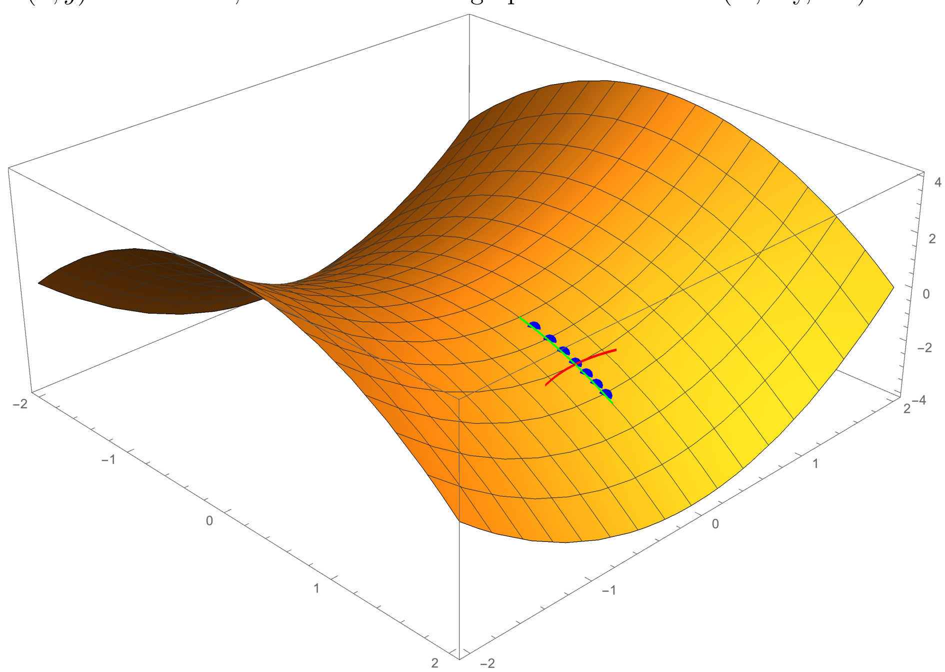
$$a(x, y) = y^2 - x^2.$$

The only constraint on b is that it couple with a to give an invertible function near $(x, y) = (1, 0)$, so we *pick* $b(x, y) = y$, so that the point $(1, 0)$ in (x, y)–coordinates corresponds to $(-1, 0)$ in (a, b)–coordinates. Back-solving, we learn that under this choice we have

$$y(a, b) = b, \qquad x(a, b) = \sqrt{b^2 - a},$$

You'll note that $x(a, b)$ is only well- defined in a range of a values for any given b and that $a(x, y)$ is only injective in a range of x and y values.

What are these coordinate transformations doing? Well, we can draw a picture of what it looks like to "move in the a–direction" or to "move in the b–direction": the segment of radius $3/4$ in the a–direction centered at $(-1, 0)$ doesn't escape the domain of definition of the coordinate transformation. We can take this segment, translate it into a curve in (x, y)–coordinates, then draw it on the graph of the function (which we have done in green). The segment of radius $1/3$ in the b–direction centered at $(-1, 0)$ doesn't escape the domain of definition of the coordinate transformation either, so we can also take this segment, translate it into a curve in (x, y)–coordinates, and draw it on the graph of the function (in, say, red). Here's the picture:



There are two things to notice: the red coordinate is *not* a linear function, but rather bends away from a straight line as we progress along the curve. It also has another property: it traces out an altitude contour of the graph, so that moving in the red direction *doesn't change the output value of f*. The green coordinate, on the other hand, has the property that it is "unit speed" as you traverse the different possibilities of the variable a. The blue dots are spaced at 1/5th intervals in a (i.e., they look like $\ldots$, $(-1/5, 0)$, $(0, 0)$, $(1/5, 0)$, $(2/5, 0)$, $\ldots$), and if you look closely you can see that they start to cluster up at they move down the parabola — because they don't have to be spread as far apart horizontally to move vertically by $1/5$ units.