Math 25 Lecture Notes
Alternating Algebra

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Definition 0.1. *A* k-tensor *on V is a multilinear function* $\underbrace{V \times V \cdots \times V}_{k \text{ times}} \to \mathbb{R}$*, and the set of k-tensors is written $T^k(V)$.*

Definition 0.2. *The* tensor product $f \otimes g$ *of a k-tensor f and an ℓ-tensor g is the $(k+\ell)$ tensor*

$$(f \otimes g)(v_1, \ldots, v_{k+\ell}) = f(v_1 \ldots v_k) \cdot g(v_{k+1} \ldots v_{k+\ell}) \tag{1}$$

Lemma 0.3. *If $u_1 \ldots u_n$ is a basis for V^*, then $u_{i_1} \ldots u_{i_k}$ is a basis for $T^k(V)$, where i_j is any sequence on $\{1 \ldots k\}$.*

Proof. Take $v_1 \ldots v_n$ dual to $u_1 \ldots u_n$. For an arbitrary tensor $t \in T^k(V)$, consider some input vectors $w_1 ... w_k$, so $w_j = \sum_{i=1}^n c_{ij} v_i$. Then

$$\begin{aligned} &t(w_1 \ldots w_j) \\ &= t(\sum_{i_1=1}^n c_{i_1 1} v_{i_1}, \ldots, \sum_{i_k=1}^n c_{i_k k} v_{i_k}) \\ &= \sum_{i_1=1}^n c_{i_1 1} t(v_{i_1}, \ldots, \sum_{i_k=1}^n c_{i_k k} v_{i_k}) \\ &\vdots \\ &= \sum_{i_1=1}^n \cdots \sum_{i_k=1}^n c_{i_1 1} \ldots c_{i_k k} t(v_{i_1} \ldots v_{i_k}) \end{aligned} \tag{2}$$

We could just as well write

$$C_{i_1, \ldots, i_k} := t(v_{i_1}, \ldots, v_{i_k}) \tag{3}$$

from which it's easy to verify

$$t = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} C_{i_1,\ldots,i_k} u_{i_1} \oplus \cdots \oplus u_{i_n} \tag{4}$$

Because we recover the $c_{i_1 1} \ldots c_{i_k k} t(v_{i_1}, \ldots, v_{i_k})$ precisely when the inputs are right.

$\square$

Example 0.4.

- *The determinant is an element of $T^{\dim V}(V)$. Special property: alternating.*
- *An inner product is a special element of $T^2(V)$. Special property: symmetric, positive definite.*
- *Finally, $T^1(V)$ is just V.*

We talked briefly about k-forms, which makes us interested in *alternating* tensors.

Definition 0.5. *For $\tau \in T^k(V)$, let*

$$\mathrm{Alt} : T^k(V) \to T^k(V)$$
$$\mathrm{Alt}(\tau)(v_1, \ldots, v_k) = \sum_{\sigma \text{ is a permutation}} \frac{1}{k!} \mathrm{sign}(\sigma) \tau(v_{\sigma(1)} \ldots v_{\sigma(k)}) \tag{5}$$

Lemma 0.6. Alt *is idempotent (i.e.* $\mathrm{Alt} \circ \mathrm{Alt} = \mathrm{Alt}$*), and has as image the alternating tensors, $\Omega^k(V)$.*

Proof. Let p a permutation swapping two indices. Since the map $\sigma \to \sigma p^{-1}$ is an isomorphism on the set of permutations on k letters, we could just as well index the sum by $\sigma' = \sigma p^{-1}$, which gets the second equality. Then

$$\mathrm{Alt}(\tau \circ p) = \frac{1}{k!} \sum_{\sigma} \mathrm{sign}(\sigma) \tau(v_{\sigma p(1)} \ldots v_{\sigma p(k)}) = \frac{1}{k!} \sum_{\sigma} \mathrm{sign}(\sigma p^{-1}) \tau_(v_{\sigma_1} \ldots v_{\sigma_k}) \tag{6}$$

And $\mathrm{sign}(\sigma p^{-1}) = -\mathrm{sign}(\sigma)$. So $\mathrm{Alt}(\tau \circ p) = -\mathrm{Alt}(\tau)$.

This argument is easily extended to general permuatations. If already $\tau = \text{Alt}(\eta)$, then

$$\begin{aligned}\text{Alt}(\tau) = \frac{1}{k!}\sum_\sigma \text{sign}(\sigma)\text{Alt}(\eta)(v_{\sigma(1)} \dots v_{\sigma(k)})\\ \frac{1}{k!}\sum_\sigma \text{sign}(\sigma)\text{Alt}(\eta \circ \sigma)\\ = \frac{1}{k!}\sum_\sigma \text{sign}(\sigma)^2\text{Alt}(\eta)\\ = \text{Alt}(\eta) = \tau\end{aligned} \tag{7}$$

□

Let's take a break and note two additional nice properties this argument gives us:

- *Sign changes when you swap two arguments*
- *If two arguments are the same, the alternating form must be zero.* Otherwise, the sign change would lead to a contradiction.

We'd like a lemma like our first about bases of the alternating tensors. For this, we need a product on alternating tensors.

Definition 0.7. *For $\tau \in T^k(V)$, $\eta \in T^\ell(V)$, define the wedge product*

$$\tau \wedge \eta = \frac{(k+\ell)!}{k!\ell!}\text{Alt}(\tau \oplus \eta) \tag{8}$$

This map is obviously bilinear, alternating. It turns out to be associative, though this is hard to prove (see Theorem 4-4. in Spivak).

Lemma 0.8. *The set of all $u_{i_1} \wedge \cdots \wedge u_{i_k}$, where $1 \le i_1 < i_2 \cdots < i_k \le \dim V := n$ is a basis for $\Omega^k(V)$.*

Proof. Write $\tau \in \Omega^k(V) \subset T^k(V)$ as $\tau = \sum_I c_I u_{i_1} \oplus \cdots \oplus u_{i_k}$. Then $\text{Alt}(\tau) = \tau$, but

$$\text{Alt}(\sum c_I u_{i_1} \oplus \dots u_{i_n}) = \sum C_I u_{i_1} \wedge \cdots \wedge u_{i_n} \tag{9}$$

If $j < k$ but $u_{i_j} > u_{i_k}$, then because the wedge product is alternating we can swap them and only accumulate a minus sign, so we need only consider $1 \le i_1 \le \cdots \le i_k \le n$. And if $i_j = i_{j'}$, then again because the wedge product is alternating that term is zero. □

Corollary 0.8.1. $\dim V = n$ *means that* $\dim \Omega^k(V) = \binom{n}{k}$.

Corollary 0.8.2. *For* $\Omega^n(V)$*, and a vector* $u_j = \sum c_{ij} v_i$*, then*

$$w(u_1, \ldots, u_n) = \det(c_{ij}) w(v_1, \ldots, v_n) \tag{10}$$

Proof. $\binom{n}{n} = 1$, and det is in the space, so anything else must be a linear multiple. □

Corollary 0.8.3. *A choice of* $w \in \Omega^n(V)$ *partitions* ***ordered bases*** *into two sets: those with* $w > 0$ *and with* $w < 0$*. A choice of preferred sign is called an* ***orientation*** *of* V *with respect to* w*. We write* $[v_1 \ldots v_n] \in \{\pm 1\}$ *for the orientation to which a given basis belongs under* w*.*

Remark 0.9. *If* V *has an inner product, there is a unique* w *for any ordered orthonormal basis* $v_1 \ldots v_n$ *so that* $w(v_1 \ldots v_n) = \mu$*, where* $\mu \in \{\pm 1\}$ *is some preferred orientation. The* w *constructed this way is called the* ***volume element****. For example,* $[e_1, \ldots, e_n] = 1$ *in* $\mathbb{R}^n$ *in* $\mathbb{R}^n$ *has determiner* det*.*

Remark 0.10. *For* $v_1, \ldots, v_{n-1} \in V$*, there is a unique* $v_n \in V$ *such that*

$$\langle w | v_n \rangle = \det \left(v_1 | \quad \ldots | \quad v_{n-1} \quad | w \right) \tag{11}$$

The vector v_n *is often called the* ***cross product*** *of* $(v_1 \ldots v_{n-1})$*. For* $n = 3$*, there is the usual mnemnonic*

$$v_1 \times v_2 = \det \begin{pmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ v_{1x} & v_{1y} & v_{1z} \\ v_{2x} & v_{2y} & v_{2z} \end{pmatrix} \tag{12}$$

One constraint is that for the determinant to be nonzero, we need linear independence: so $\langle v_n, v_n \rangle$ constrains v_n to a one-dim subspace, if the $v_1 \ldots v_{n-1}$ are linearly independent. Now you can just plug in some values to get the right linear multiple.