QUADRATIC FORMS

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From a quadratic form, say $Q(\mathbf{x}) = \sum_{1\le i\le j\le n} a_{ij}x_ix_j$ for $\mathbf{x} = [x_1, \dots, x_n]$, let $M_Q = \{m_{ij}\}$ be the n by n matrix given by $m_{ii} = a_{ii}$, and $m_{ij} = m_{ji} = \frac{a_{ij}}{2}$ for $1 \le i < j \le n$. This matrix is set up to have two properties.

- M_Q is symmetric.
- $\mathbf{x}M_Q\mathbf{x}^T = Q(\mathbf{x})$.

For example, if $Q(\mathbf{x}) = x_1^2 + 3x_1x_2 + 2x_2^2$, then $M_Q = \begin{bmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{bmatrix}$.

By the first property, and by the Spectral Theorem, M_Q has real eigenvalues and is diagonalizable by an orthogonal matrix. That is, there is an orthogonal matrix P (so $P^T = P^{-1}$) and a real diagonal matrix D with $M_Q = PDP^{-1}$. Since P is orthogonal, substitution yields $M_Q = PDP^T$.

Thus $Q(\mathbf{x}) = \mathbf{x}M_Q\mathbf{x}^T = \mathbf{x}PDP^T\mathbf{x}^T = (\mathbf{x}P)D(\mathbf{x}P^T)$. Now, let the diagonal entries of D be $\lambda_1, \dots, \lambda_n$, and the columns of P be $v_1, \dots, v_n$. Then $\mathbf{x}P = [\mathbf{x}v_1, \dots, \mathbf{x}v_n]$. So substituting, we have that

$$Q(\mathbf{x}) = [\mathbf{x}v_1 \quad \cdots \quad \mathbf{x}v_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{x}v_1 \\ \vdots \\ \mathbf{x}v_n \end{bmatrix} = \sum_{i=1}^{n} \lambda_i(\mathbf{x}v_i)^2.$$

But $\mathbf{x}v_i$ is some linear expression in the x_i's, so we have just written $Q(\mathbf{x})$ as a weighted sum of squares! We can pull the (absolute value of) λ_i inside the square, to obtain

$$Q(\mathbf{x}) = \sum_{i=1}^{n} \operatorname{sgn}(\lambda_i)\left(\sqrt{|\lambda_i|}\mathbf{x}v_i\right)^2.$$

Now, we have written $Q(\mathbf{x})$ in our standard form–the sum and difference squares of linear functions of the x_i's! We know the functions are linearly independent because the v_i are linearly independent (this is what it means for the matrix M_Q to be diagonalizable, which it is by the Spectral Theorem). Thus we can read out the signature exactly from this expression! The number of positive (respectively negative) terms will just be the number of positive (respectively negative) eigenvalues of M_Q.

Date: 3/30/2017.