A FUN FUBINI INTEGRAL
SECTION 5, MATH 25B

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Problem. Find $\displaystyle\int_0^\infty \frac{\tan^{-1}(2x)-\tan^{-1}(x)}{x}$.

Solution. The idea is to write this as a double integral, and then switch the order of integration per Fubini.

Now, note that since $\frac{\partial}{\partial y}\tan^{-1}(xy) = \frac{x}{1+x^2y^2}$, by our usual 1-dimensional Fundamental Theorem of Calculus, we have that

$$\int_1^2 \frac{x}{1+x^2y^2}dy = \tan^{-1}(xy)\Big|_{y=1}^2 = \tan^{-1}(2x)-\tan^{-1}(x).$$

Since x is a constant in the integral to the left, we can divide through by it to obtain

$$\frac{\tan^{-1}(2x)-\tan^{-1}(x)}{x} = \int_1^2 \frac{1}{1+x^2y^2}dy.$$

Thus our original integral can be written as

$$\int_0^\infty \int_1^2 \frac{1}{1+x^2y^2}dydx.$$

So we are taking a double integral over the region $[0,\infty]\times[1,2]$. We can now switch the order of integration by Fubini (in a serious context, we'd first have to make sure this is integrable but this follows from some bounding) to say that this is in fact equal to

$$\int_1^2 \int_0^\infty \frac{1}{1+x^2y^2}dxdy.$$

Now we can evaluate the inner integral. In fact, we have already evaluated the integral of $\frac{1}{1+x^2y^2}$ (with respect to y, but it's symmetric!) and so

$$\int_0^\infty \frac{1}{1+x^2y^2}dx = \frac{1}{y}\tan^{-1}(xy)\Big|_{x=0}^\infty = \frac{\pi}{2y}.$$

(Technically if y is negative it would be $-\frac{\pi}{2y}$ as $\tan^{-1}(-\infty) = -\frac{\pi}{2}$ but y is always in $[1,2]$ in our case.) Therefore, the answer simply becomes

$$\int_1^2 \frac{\pi}{2y}dy = \frac{\pi}{2}\log y\Big|_{y=1}^2 = \frac{\pi\log 2}{2}.$$

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