EXPANDING ON DIFFERENTIATING MATRIX INVERSION
SECTION 3, MATH 25B

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We showed in class that the map $\chi : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ given by $A \to A^{-1}$ has derivative $(D_A\chi)H = -A^{-1}HA^{-1}$.

So what does this expression mean? Why does it make sense? What type of answer would make sense here? Well, $D_A\chi$ should be some linear operator at every point A. It should send directions–given by a vector in our space, which is in this case an n by n matrix H–to the directional derivative of χ in that direction, which should be some vector in our image space–which is again some n by n matrix. So $D_A\chi$ should be some linear map sending n by n matrices to other n by n matrices.

Indeed, $H \to -A^{-1}HA^{-1}$ is a linear map sending n by n matrices to other n by n matrices. (It is linear as both left- and right-multiplication are linear.) This case is a bit weird since the most natural way of depicting $D_A(\chi)$ is not with the matrix of partials, which is what we normally use. It turns out that just seeing the matrix as a linear map, as in the definition, is very nice here. (If we were to use a matrix, it would need dimensions of n^2 on each side, since that's the dimension of our base space.)

In class we proved this by considering the composition of maps $GL_n(\mathbb{R}) \to \mathrm{Mat}_n(\mathbb{R})^2 \to \mathrm{Mat}_n(\mathbb{R})$, where the first map is given by $A \to (A, A^{-1})$, and the second map is the multiplication map $\mu : (A, B) \to AB$. Since we are looking for (the second component of) the derivative of the first map, and we know that the composition of these maps is the identity, we can use the chain rule to solve for what we are looking for. The argument is similar to solving for the derivative of $\frac{1}{f}$ by saying that $f \cdot \frac{1}{f} = 1$, so taking the derivative of both sides and using the product rule, $\frac{f'}{f} + f\left(\frac{1}{f}\right)' = 0$, so $\left(\frac{1}{f}\right)' = \frac{f'}{f^2}$. The tricky aspects are that we now have a multivariable function, and our product now does not commute (as it's matrix multiplication).

Here's another way (in a similar spirit) to compute the same thing. Let's go back to the definition of the derivative. We know that we should have

$$\lim_{M \to 0} \frac{\chi(A + M) - \chi(A) - (D_A\chi)(M)}{|M|} = 0.$$

(We have to ask ourselves what $|H|$ means–what does it mean to take the norm of a matrix? Normally this notation means determinant, but that's not what we mean here. What we're really doing is identifying the space of n by n matrices with $\mathbb{R}^{n^2}$, so it should just be the square root of the sum of the squares of all the entries of the matrix H. This doesn't matter too much, all that matters is that it scales linearly; that is, $|tH| = t|H|$.)

But $\chi(A + H) = (A + H)^{-1}$, and similarly $\chi(A) = A^{-1}$ by definition. There's something here we have to watch out for–how do we know that $A + H$ is invertible? Well, the determinant is a continuous function, and $\mathbb{R} - \{0\}$ is an open set, so the preimage of that under the determinant map must be open. Thus the set of invertible matrices is open, so since A is invertible, for sufficiently small H, $A + H$ will be invertible, so we're fine on this aspect.

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Now, one way of letting $H \to 0$ is to instead replace M by tH for some fixed matrix M and let $t \to 0$. So we obtain

$$0 = \lim_{t \to 0} \frac{\chi(A + tH) - \chi(A) - (D_A\chi)(tH)}{|tH|} = \lim_{t \to 0} \frac{\chi(A + tH) - \chi(A) - t(D_A\chi)(H)}{t|H|}.$$

Since $|H|$ is now fixed with respect to the limit, we can multiply through by it. Cancelling out t from the numerator and denominator, we obtain

$$\lim_{t \to 0} \left(\frac{\chi(A + tH) - \chi(A)}{t} - (D_A\chi)(H) \right) = 0,$$

so

$$(D_A\chi)(H) = \lim_{t \to 0} \frac{\chi(A + tH) - \chi(A)}{t}.$$

The work above is exactly the derivation that the directional derivative is exactly the total derivative at a particular vector, which we know. Now, we need to compute the limit. Substituting in the definition of χ,

$$(D_A\chi)(H) = \lim_{t \to 0} \frac{(A + tH)^{-1} - \chi(A)^{-1}}{t}.$$

Now,

$$\begin{aligned} (A + tH)^{-1} &= A^{-1} \left(A^{-1}\right)^{-1} (A + tH)^{-1} \\ &= A^{-1} \left((A + tH)A^{-1}\right)^{-1} \\ &= A^{-1} \left(I + tHA^{-1}\right)^{-1}, \end{aligned}$$

so we can pull out an A^{-1} on the left in the expression above to obtain

$$(D_A\chi)(H) = A^{-1} \lim_{t \to 0} \frac{\left(I + tHA^{-1}\right)^{-1} - I}{t}.$$

Now, the question becomes, what is $\left(I + tHA^{-1}\right)^{-1}$? Well, note that the identity

$$(1 - r)^{-1} = 1 + r + r^2 + \cdots$$

is true regardless of whether r is actually a matrix! The derivation works exactly the same, as r commutes with all of its powers. The only caveat is that the right side must converge, which will be true if the entries of r are sufficiently small. Thus for t sufficiently small, we can substitute $r = -tHA^{-1}$ in this expression to obtain

$$\left(I + tHA^{-1}\right)^{-1} = I + tHA^{-1} + t^2 \left(HA^{-1}\right)^2 + t^3 \left(HA^{-1}\right)^3 + \cdots,$$

so when substituting this in we obtain

$$\begin{aligned} (D_A\chi)(H) &= A^{-1} \lim_{t \to 0} \frac{\left(I + tHA^{-1}\right)^{-1} - I}{t} \\ &= A^{-1} \lim_{t \to 0} \frac{\left(I + tHA^{-1} + t^2 \left(HA^{-1}\right)^2 + t^3 \left(HA^{-1}\right)^3 + \cdots\right) - I}{t} \\ &= A^{-1} \lim_{t \to 0} \frac{tHA^{-1} + t^2 \left(HA^{-1}\right)^2 + t^3 \left(HA^{-1}\right)^3 + \cdots}{t} \\ &= A^{-1} \lim_{t \to 0} HA^{-1} + t \left(HA^{-1}\right)^2 + t^2 \left(HA^{-1}\right)^3 + \cdots \\ &= A^{-1}HA^{-1}, \end{aligned}$$

finishing our derivation.