Math 25b Lecture Notes

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1 More Topology on $\mathbb{R}$

Last time, we proved:

Theorem 1.1. *(Heine-Borel) The set $[a, b]$ is compact.*

This had the following corollary:

Corollary 1.2. *(Bolzano-Weierstrass) Every sequence inside of a compact set has a convergent subsequence.*

However, note that in the proof of Bolzano-Weierstrass we actually used the *converse* to the theorem.

Lemma 1.3. *A compact subset of $\mathbb{R}$ decomposes into a finite union of closed intervals.*

We will actually first show the following statement.

Lemma 1.4. *Compact sets are closed and bounded.*

Proof. Suppose X is compact.

First, we show X is bounded. Consider the open intervals $U_n = (-n, n)$.

Since the sets U_n cover the real line, we find that the sets U_n cover X. Since X is compact, there exists $i_1, \ldots, i_k$ such that $U_{i_1}, \ldots, U_{i_k}$ cover X. Therefore, if we set $M = \max(i_1, \ldots, i_k)$, by definition U_M covers X. Since $U_M = (-M, M)$ we find that X is bounded.

Next, we show X is closed. Suppose for the sake of contradiction that it is not closed. Then, $\mathbb{R} - X$ is not open, so there exists a point $s \notin X$ such that for every $\varepsilon > 0$ the open interval $(s - \varepsilon, s + \varepsilon)$ has nonempty intersection with X.

Then, we can pick open sets $U_n = (-\infty, s - 1/n)$ and $V_n = (s + 1/n, \infty)$. The union of these sets is $\mathbb{R} - \{s\}$, so they form a cover of X.

Note that any finite subset of these will at best cover $\mathbb{R} - [s - 1/N, s + 1/N]$ for some large N. Therefore, there is no finite subcover and we arrive at a contradiction.

■

Next, we will prove the converse.

Lemma 1.5. *Closed subsets of compact sets are compact.*

Proof. Let V be a closed subset of a compact set X. Suppose that $\{U_\alpha\}$ is an open cover of V.

Since V is closed, $\mathbb{R} - V$ is open. Then, the collection $\{U_\alpha\} \cup \mathbb{R} - V$ is an open cover of X. This has a finite subcover by compactness of X. This finite subcover covers V, and is contained in the cover $\{U_\alpha\}$ of V if and only if it does not contain $\mathbb{R} - V$. However, $\mathbb{R} - V$ has zero intersection with V, so removing it from the subcover gives us a finite subcover of V. ■

This allows us to show:

Lemma 1.6. *Closed and bounded subsets are compact.*

Proof. If X is bounded, then we have X is contained in some closed interval $[-N, N]$ for $N \in \mathbb{R}$. By Heine-Borel, $[-N, N]$ is compact. Since X is now a closed subset of a compact set, it is compact by our lemma. ■

We proceed to define the topological concept of connectedness.

Definition 1.7. A **disconnect** of a subset $X \subseteq \mathbb{R}$ is a pair of nonempty open sets U, V such that X is contained in their union and $U \cap V = \emptyset$. A set is called **connected** if it does not admit a disconnect.

Lemma 1.8. *A connected subset $X \subseteq \mathbb{R}$ takes the form of an interval.*

Proof. We will show that if X "fails to be an interval" then it is not connected.

We make this rigorous by saying that X "fails to be an interval" if we can find $a, b \in X$ and another point y such that $a \leq y \leq b$ and $y \notin X$.

Set $U = (-\infty, y)$ and $V = (y, \infty)$. These form a disconnect of X, so X is not connected. ■

Next we combine this notion with our earlier discussion on closedness and boundedness.

Lemma 1.9. *Closed bounded connected subsets in $\mathbb{R}$ are closed intervals.*

Proof. Since X is bounded, it has an infimum $\inf X$ and a supremum $\sup X$. Since X is closed, these belong to X.

Because X is connected, by the above lemma it must be equal to $[\inf X, \sup X]$. ■

Corollary 1.10. *A compact subset of $\mathbb{R}$ decomposes as a finite union of disjoint closed, bounded intervals.*

Proof. To be completed later. ■