Homework #8

Math 25b

Due: April 26th, 2017

Guidelines:

- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one, a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!)[1]

1 For submission to Thayer Anderson

Problem 1.1. Let $g\colon A \to \mathbb{R}^p$ be a differentiable function defined on an open $A \subseteq \mathbb{R}^n$, such that Dg is of rank p everywhere on $M = g^{-1}(0)$, which we then know to be a manifold. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be an auxiliary differentiable function which we hope to optimize on M. Show that if a maximum or minimum of f on M occurs at $a \in M$, show that there are $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$ such that

$$\frac{\partial f}{\partial x_j}(a) = \sum_{i=1}^{p} \lambda_i \frac{\partial g_i}{\partial x_j}(a).$$

Start by giving a geometric interpretation of this condition.

Problem 1.2. You can try to use Problem 1.1 to solve for such points a: the system of equations involving λ give n equations in $(n+1)$ unknowns, and then the restriction $g(a) = 0$ gives an additional equation. Try to apply this idea in the following problem:

1. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be self-adjoint under the usual inner product, and suppose that in the usual basis it takes the matrix form $A = (a_{ij})$, so that $a_{ij} = a_{ji}$. Set $f(x) = \langle Tx, x\rangle$, and show $D_k f(x) = 2\sum_{j=1}^{n} a_{kj}x^j$. By consider the maximum of $\langle Tx, x\rangle$ on S^{n-1}, show that there is $x \in S^{n-1}$ and $\lambda \in \mathbb{R}$ such that $Tx = \lambda x$.
2. If $V = \{y \in \mathbb{R}^n \mid \langle x, y\rangle = 0\}$, show that $T(V) \subseteq V$ and $T\colon V \to V$ is self-adjoint.
3. Show that T has a basis of eigenvectors.

[1] This version of the homework dates from April 25, 2017.

2 For submission to Davis Lazowski

Problem 2.1.

1. In class, we claimed that the zero-locus of a sufficiently nice function formed a manifold. Show a partial converse to this: for $M \subseteq \mathbb{R}^n$ a k–manifold and $x \in M$ a point on it, show there exists an open neighborhood $A \subseteq \mathbb{R}^n$ of x and a differentiable function $g\colon A \to \mathbb{R}^{n-k}$ such that $g^{-1}(0) = A \cap M$ and the derivative of g is of rank $(n-k)$ on this locus.
2. If $M \subseteq \mathbb{R}^n$ is an orientable $(n-1)$–manifold, show that there is an open set $A \subseteq \mathbb{R}^n$ and a differentiable function $g\colon A \to \mathbb{R}$ so that $M = g^{-1}(0)$ and g has nonvanishing derivative on M. (This globalizes the previous problem: use orientation and partitions of unity to sew together the local solutions.)

Problem 2.2. Suppose that $M \subseteq \mathbb{R}^n$ is a compact $(n-1)$–manifold, and let M_ε be the following set of points:

$$M_\varepsilon = \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{c} \text{there is a } y \in M \text{ such that } x = y \pm \varepsilon n_y, \\ \text{where } n_y \text{ is the normal vector to } M \text{ at } y \end{array} \right\}.$$

1. Show that ε can be taken small enough so that M_ε is also a manifold.
2. Sketch what M_ε looks like for the Möbius band. Is the resulting manifold orientable?
3. Inspired by this, show in general that M_ε is always *orientable*, even if M is not.

3 For submission to Handong Park

Problem 3.1. Show that a tangent space of a manifold T_xM consists exactly of tangent vectors $(D_0\gamma)(1)$ where $\gamma\colon (-\varepsilon, \varepsilon) \to M$ is a curve in M with $\gamma(0) = x$.

Problem 3.2. Show that Stokes's theorem for manifolds can fail if the manifold is not compact. (Hint: find a manifold M that uses noncompactness to achieve $\partial M = 0$.)[2]

Problem 3.3. In the course of solving Practice Midterm #2.2, you found a way to (recursively) express the volume of the unit ball in $\mathbb{R}^n$. Use the divergence theorem to relate the volume of the unit ball in $\mathbb{R}^n$ to the $(n-1)$–dimensional area of the unit sphere in $\mathbb{R}^n$. You will probably want to make use of the $(n-1)$–form

$$((v_1, \ldots, v_{n-1}) \in T_x\mathbb{R}^n) \mapsto \det(v_1 | \cdots | v_{n-1} | x).$$

4 For submission to Rohil Prasad

Problem 4.1. Consider the element $\omega \in \Omega^2\mathbb{R}^3$ defined by

$$\omega = \frac{x \,\mathrm{d}y \wedge \mathrm{d}z + y \,\mathrm{d}z \wedge \mathrm{d}x + z \,\mathrm{d}x \wedge \mathrm{d}y}{(x^2 + y^2 + z^2)^{3/2}}.$$

1. Show that ω is closed.
2. Let $S_r = \{v \in \mathbb{R}^3 : \|v\| = r\}$ be the sphere of length r vectors, a 2–manifold. Verify the formula

$$\omega_p(h_1, h_2) = \frac{\langle h_1 \times h_2, p \rangle}{\|p\|^3},$$

and conclude that ω restricted to S_r is r^{-2} times the volume element.

[2] If you're feeling really feisty, you should spend a little bit imagining how you could modify our set-up to eliminate this behavior. The problem is that your example M *wants* to have boundary, but the place where the boundary belongs escapes $\mathbb{R}^n$. How could you modify $\mathbb{R}^n$ itself?

3. Make the calculation $\int_{S_r} \omega = 4\pi$, and conclude that ω is not exact. This element is the analogue of $d\theta$ in $\mathbb{R}^2 \setminus \{0\}$, so we re-notate ω as $d\Theta$.

4. Let $p \in \mathbb{R}^3$ be any point and let $h \in T_p\mathbb{R}^3$ be a tangent vector collinear the origin, i.e., $h = \lambda p$ for some $\lambda \in \mathbb{R}$. Show $d\Theta_p(h, h') = 0$ for *any* $h' \in T_p\mathbb{R}^3$. Defining a *generalized cone* to be a manifold which is a union of rays through the origin (cf. N in Figure 5-10 in the book), show that $d\Theta$ integrated over a generalized cone always gives 0.

5. Suppose a manifold M has the property that every ray through the origin intersects M exactly once. Define the *generalized cone through* M, $C(M)$, to be the collection of these rays. The *solid angle subtended by* M is defined to be the area of $C(M) \cap S_1$ (or, equivalently, r^{-2} times the area of $C(M) \cap S_r$ for any r). Prove that the solid angle subtended by M can be computed by

$$\int_M \mathrm{d}\Theta.$$

(Note that this integral does *not* have a cone in it.) (Again, look at Figure 5-10 for a clue.)