Homework #8 Solutions

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1 For submission to Thayer Anderson

Problem 1.1. Let $g : A \to \mathbb{R}^p$ be a differentiable function defined on an open $A \subset \mathbb{R}^n$, such that Dg is of rank p everywhere on $M = g^{-1}(0)$, which we then know to be a manifold. Let $f : \mathbb{R}^\ltimes \to \mathbb{R}$ be an auxiliary differentiable function which we hope to optimize on M. Show that if a maximum or minimum of f on M occurs at $a \in M$, show that there are $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$ such that

$$\frac{\partial f}{\partial x_j}(a) = \sum_{i=1}^{p} \lambda_i \frac{\partial g_i}{\partial x_j}(a)$$

Start by giving a geometric interpretation of this condition.

Solution. The geometric statement of this claim is that the gradient of f must be perpendicular to the manifold, M, at the the point a. Certainly, this claim must be true, or we could project $\nabla f(a0$ along M and move in that direction to increase or decrease the value of f, which would be a contradiction. Now we will justify why the problem is equivalent to this necessarily true geometric claim.

Since M is a level curve of g, it follows that it is a level curve in each component, $g_1, g_2, \ldots g_p$. Thus each ∇g_i is perpendicular to M. Furthermore, because the matrix $(D_a g)' = \{\nabla g_1 | \nabla g_2 | \cdots | \nabla g_p\}$ has full rank by assumption, it follows that the ∇g_i are linearly independent. Morever it follows that they span the orthogonal complement of the manifold. Thus they form a basis. The claim

$$\frac{\partial f}{\partial x_j}(a) = \sum_{i=1}^{p} \lambda_i \frac{\partial g_i}{\partial x_j}(a)$$

can be therefore be interpreted to mean exactly that $\nabla f \in \operatorname{span}\{g_1, \ldots, g_p\}$, which is the geometric claim that I have already argued. (TA)

Problem 1.2. You can try to use Problem 1.1 to solve for such points a: the system of equations involving λ give n equations in $(n+1)$ unkowns and then the restriction $g(a) = 0$ gives an additional equation. Try to apply this idea in the following problem:

1. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be self-adjoint under the usual inner product, and suppose that in the usual basis it takes the matrix form $A = (a_{ij})$, so that $a_{ij} = a_{ji}$. Set $f(x) = \langle Tx, x\rangle$ and show that $D_k f(x) = 2\sum_{j=1}^n a_{kj} x^j$. By considering the maximum of $\langle Tx, x\rangle$ on S^{n-1}, show that there is $x \in S^{n-1}$ and $\lambda \in \mathbb{R}$ such that $Tx = \lambda x$.
2. If $V = \{y \in \mathbb{R}^n | \langle x, y\rangle = 0\}$, show that $T(V) \subset V$ and $T : V \to V$ is self-adjoint.
3. Show that T has a basis of eigenvectors.

Solution. 1. We can compute $f(x)$ by first computing Tx:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \vdots \\ \dots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 & \vdots & x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \dots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n \end{pmatrix}$$

Then $f(x)$ is given by

$$f(x) = \langle T(x), x \rangle = \sum_{j=1}^{n} \left(x_j \sum_{i=1}^{n} a_{ji}x_i \right)$$

To calculate $D_k f(x)$ we take the partial derivative by x_k. This kills all the terms without an x_k (that is, where i and j are not equal to k):

$$\begin{aligned} &\frac{\partial}{\partial x_k} \sum_{j \neq k} \left(x_j \sum_{i \neq k} a_{ji}x_i \right) + \frac{\partial}{\partial x_k} \sum_{j=k} \left(x_j \sum_{i \neq k} a_{ji}x_i \right) + \frac{\partial}{\partial x_k} \sum_{j \neq k} \left(x_j \sum_{i=k} a_{ji}x_i \right) + \frac{\partial}{\partial x_k} a_{kk}x_k^2 \\ &= 0 + \frac{\partial}{\partial x_k} \left(\sum_{i \neq k} a_{ki}x_i x_k \right) + \frac{\partial}{\partial x_k} \left(\sum_{j \neq k} a_{jk}x_j x_k \right) + \frac{\partial}{\partial x_k} a_{kk}x_k^2 \end{aligned}$$

Due to self adjointness ($a_{ki} = a_{ik}$) we see that these the first two terms contribute equally, we and we get

$$\frac{\partial f(x)}{\partial x_k} = \sum_{i=1}^{n} 2a_{ki}x_i$$

Since S^{n-1} is compact and f continuous, we can find a maximum of f on S^{n-1}. If we can construct a function, $g : \mathbb{R}^n \to \mathbb{R}$ with derivative of rank 1 such that $S^{n-1} = g^{-1}(0)$ then we will be guaranteed a λ that satisfies the problem description. The function ($g(x) = x_1^2 + \dots + x_n^2 - 1$ is certainly such a function. We see that $Dg = (2x_1 \ \dots \ 2x_n)$. Then at the maximum point

$$\frac{\partial f(x)}{\partial x_k} = \sum_{i=1}^{n} 2a_{ki}x_i = 2\lambda x_k$$

Thus $T(x) = \lambda x$.

Take fixed x as in the previous part that maximizes $\langle Tx, x \rangle$ on S^{n-1}. Then, by self-adjointness

$$\langle x, Ty \rangle = \langle Tx, y \rangle = \langle \lambda x, y \rangle = \lambda \langle x, y \rangle = 0$$

Therefore $T(V) \subset V$. It follows that the induced operator $T : V \to V$ is self-adjoint because T is self-adjoint. By the spectral theorem for self-adjoint operators, T admits a basis of eigenvectors. (TA)

2 For submission to Davis Lazowski

Problem 2.1. 1. In class, we claimed that the zero-locus of a sufficiently nice function formed a manifold. Show a partial converse to this: for $M \subseteq \mathbb{R}^n$ a k–manifold and $x \in M$ a point on it, show there exists an open neighborhood $A \subseteq \mathbb{R}^n$ of x and a differentiable function $g\colon A \to \mathbb{R}^{n-k}$ such that $g^{-1}(0) = A \cap M$ and the derivative of g is of rank $(n-k)$ on this locus.

2. If $M \subseteq \mathbb{R}^n$ is an orientable $(n-1)$–manifold, show that there is an open set $A \subseteq \mathbb{R}^n$ and a differentiable function $g\colon A \to \mathbb{R}$ so that $M = g^{-1}(0)$ and g has nonvanishing derivative on M. (This globalizes the previous problem: use orientation and partitions of unity to sew together the local solutions.)

Solution. 1. This comes right out of the first local model definition of a manifold: there is an open neighborhood U of x, an open neighborhood V of 0 in $\mathbb{R}^n$, and a diffeomorphism $f\colon U \to V$ with $f^{-1}(M) = U \cap (\mathbb{R}^k \times 0)$. The map g is the composite of f^{-1} with $\mathbb{R}^n \to 0 \times \mathbb{R}^{n-k}$, and its derivative has rank $(n-k)$ *everywhere* because f^{-1} is of full rank *everywhere*.

2. Cover M by consistently oriented local models $\{U\} = \mathcal{O}$, and apply the previous problem to construct a candidate function g_U on each of them. Choose a partition of unity Φ subordinate to $\mathcal{O}$, and set $g = \sum_U \varphi_U \cdot g_U$. The orientation guarantees that the zero locus of g is as desired: wherever the open sets overlap, the relevant functions g_U all take the same sign, so their sum cannot cancel. (ECP)

Problem 2.2. Suppose that $M \subseteq \mathbb{R}^n$ is a compact $(n-1)$–manifold, and let M_ε be the following set of points:

$$M_\varepsilon = \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{c} \text{there is a } y \in M \text{ such that } x = y \pm \varepsilon n_y, \\ \text{where } n_y \text{ is the normal vector to } M \text{ at } y \end{array} \right\}.$$

1. Show that $\varepsilon > 0$ can be taken small enough so that M_ε is also a manifold.

2. Sketch what M_ε looks like for the Möbius band. Is the resulting manifold orientable?

3. Inspired by this, show in general that M_ε is always *orientable*, even if M is not.

Solution. 1. Cover M by local models such that the normal vectors in the domain of any given local model subtend a cone strictly smaller than $\pi/2$ radians. Now let ε be smaller than one third of the distance from the center point in any local model to all other points on the manifold not in this model—a nonzero number by compactness. This choice of ε guarantees that each point $x \in M_\varepsilon$ can be *uniquely* written as $x = y \pm \varepsilon n_y$ for some choice of sign and point $y \in M$. Then, each local model on M gives rise to a pair of local models on M_ε, by postcomposing g with either the function $y \mapsto y + \varepsilon n_y$ or the function $y \mapsto y - \varepsilon n_y$.

2. The resulting manifold is orientable: it is a Möbius band with two twists in it, which is, in fact, diffeomorphic to a cylinder.

3. Give the charts on M_ε the orientation which faces *away* from M. This is consistent. (ECP)

3 For submission to Handong Park

Problem 3.1. Show that a tangent space of a manifold T_xM consists exactly of tangent vectors $(D_0\gamma)(1)$ where $\gamma\colon (-\varepsilon, \varepsilon) \to M$ is a curve in M with $\gamma(0) = x$.

Solution. Since T_xM is *defined* using the local model, we solve this problem using that. A tangent vector $v \in T_xM$ is defined as the image of a tangent vector $v' \in T_0\mathbb{R}^n$ under a local model map f which compares a neighborhood of 0 in $\mathbb{R}^n$ with a neighborhood of x in $\mathbb{R}^n$ and which carries exactly the hyperplane cut out by the first k coordinates to M. Accordingly, v' must lie in the same hyperplane, and so the linear equation $\ell(t) = v' \cdot t$ gives a curve in the local model with the right tangent vector which postcomposes to give a curve $f \circ \ell$ to give a curve in M with the right tangent vector. (ECP)

Problem 3.2. Show that Stokes's theorem for manifolds can fail if the manifold is not compact. (Hint: find a manifold M that uses noncompactness to achieve $\partial M = 0$.)

Solution. Consider any nonnegative smooth function $f\colon \mathbb{R} \to \mathbb{R}$ which is nonvanishing somewhere, and let F be its antiderivative. Its integral is nonvanishing, but the boundary of $\mathbb{R}$ is $\varnothing$, hence we have a mismatch

$$\int_{\mathbb{R}} f \neq \int_{\partial\mathbb{R}} F = \int_{\varnothing} F = 0. \qquad \text{(ECP)}$$

Problem 3.3. In the course of solving Practice Midterm #2.2, you found a way to (recursively) express the volume of the unit ball in $\mathbb{R}^n$. Use the divergence theorem to relate the volume of the unit ball in $\mathbb{R}^n$ to the $(n-1)$–dimensional area of the unit sphere in $\mathbb{R}^n$. You will probably want to make use of the $(n-1)$–form

$$((v_1, \ldots, v_{n-1}) \in T_x\mathbb{R}^n) \mapsto \det(v_1|\cdots|v_{n-1}|x).$$

Solution. Write $\mathrm{d}A$ for the $(n-1)$–form in the hint, which is a volume form on S^{n-1}, the unit sphere in $\mathbb{R}^n$. Considered as an $(n-1)$–form on $\mathbb{R}^n$, it also has the property

$$\mathrm{d}\,\mathrm{d}A = n\,\mathrm{d}V,$$

where $\mathrm{d}V$ is the standard volume form on $\mathbb{R}^n$. We thus apply Stokes's theorem:

$$n\int_{B^n} \mathrm{d}V = \int_{B^n} n\,\mathrm{d}V = \int_{B^n} \mathrm{d}\,\mathrm{d}A = \int_{\partial B^n} \mathrm{d}A = \int_{S^{n-1}} \mathrm{d}A. \qquad \text{(ECP)}$$

4 For submission to Rohil Prasad

Problem 4.1. Consider the element $\omega \in \Omega^2\mathbb{R}^3$ defined by

$$\omega = \frac{x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y}{(x^2+y^2+z^2)^{3/2}}.$$

1. Show that ω is closed.
2. Let $S_r = \{v \in \mathbb{R}^3 : \|v\| = r\}$ be the sphere of length r vectors, a 2–manifold. Verify the formula

   $$\omega_p(h_1, h_2) = \frac{\langle h_1 \times h_2, p\rangle}{\|p\|^3},$$

   and conclude that ω restricted to S_r is r^{-2} times the volume element.
3. Make the calculation $\int_{S_r} \omega = 4\pi$, and conclude that ω is not exact. This element is the analogue of $d\theta$ in $\mathbb{R}^2 \setminus \{0\}$, so we re-notate ω as $d\Theta$.
4. Let $p \in \mathbb{R}^3$ be any point and let $h \in T_p\mathbb{R}^3$ be a tangent vector collinear the origin, i.e., $h = \lambda p$ for some $\lambda \in \mathbb{R}$. Show $d\Theta_p(h, h')$ for *any* $h' \in T_p\mathbb{R}^3$. Defining a *generalized cone* to be a manifold which is a union of rays through the origin (cf. N in Figure 5-10 in the book), show that $d\Theta$ integrated over a generalized cone always gives 0.
5. Suppose a manifold M has the property that every ray through the origin intersects M exactly once. Define the *generalized cone through* M, $C(M)$, to be the collection of these rays. The *solid angle subtended by* M is defined to be the area of $C(M) \cap S_1$ (or, equivalently, r^{-2} times the area of $C(M) \cap S_r$ for any r). Prove that the solid angle subtended by M can be computed by

   $$\int_M \mathrm{d}\Theta.$$

   (Note that this integral does *not* have a cone in it.) (Again, look at Figure 5-10 for a clue.)

Solution. 1. This is a matter of direct calculation:

$$\mathrm{d}\omega = \left(\frac{(x^2+y^2+z^2) - 3x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{(x^2+y^2+z^2) - 3y^2}{(x^2+y^2+z^2)^{5/2}} + \frac{(x^2+y^2+z^2) - 3z^2}{(x^2+y^2+z^2)^{5/2}} \right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = 0.$$

2. The numerator of this fraction admits an expression by a determinant: $\langle h_1 \times h_2, p \rangle = \det(h_1 \mid h_2 \mid p)$. On the other hand, the behavior of the standard 2–forms in the definition of ω can be arranged as follows:

$$\omega_p(h_1, h_2) = \frac{1}{\|p\|^3} \det \left(\begin{array}{c|c|c} p_x & & \\ p_y & h_1 & h_2 \\ p_z & & \end{array} \right).$$

Meanwhile, the definition of the volume element is to set the first column of the determinant form to a surface normal — and $p/\|p\|$ is such a vector for the sphere. The other two factors of $\|p\|$ give the final expression $\omega = r^{-2}\,\mathrm{d}V$.

3. The integral of $\mathrm{d}V$ over a surface computes its surface area, which for a 2–sphere of radius r is $4\pi r^2$. Dividing out by r^2, we indeed compute

$$\int_{S_r} \omega = \int_{S_r} \frac{1}{r^2}\,\mathrm{d}V = \frac{1}{r^2} \cdot 4\pi r^2 = 4\pi.$$

4. The tangent space to a generalized cone contains the direction $p/\|p\|$. This repeated column of the determinant causes it to vanish.

5. Stokes's theorem applied to this generalized cone shows $\int_{C(M)} \mathrm{d}\,\mathrm{d}\Theta = \int_{C(M)} 0 = 0$ to decompose as the sum of three components: the surface integral of $\mathrm{d}\Theta$ over M, the surface integral of $\mathrm{d}\Theta$ over $C(M) \cap S_1$, and the surface integral of $\mathrm{d}\Theta$ over $C(\partial M)$. The last surface integral vanishes because it is a 2–dimensional generalized cone, as in the previous part, and hence we conclude

$$\int_M \mathrm{d}\Theta = \int_{C(M) \cap S_1} \mathrm{d}\Theta. \tag{ECP}$$