Homework #7

Math 25b

Due: April 19th, 2017

Guidelines:

- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one, a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!)[1]

1 For submission to Thayer Anderson

Problem 1.1. Let $R_{\text{out}} > R_{\text{in}} > 0$ be the positive radii of two concentric circles in $\mathbb{R}^2$, both centered at the origin. Call the circles C_{out} and C_{in}. Construct a 2–chain σ with $\partial\sigma = C_{\text{out}} - C_{\text{in}}$.

Problem 1.2. Again fixing a radius R, let C_R be the circle of radius R centered at the origin.

1. Show $\int_{C_R} \mathrm{d}\theta = 2\pi$, independent of R.
2. Conclude that there is no 2–chain σ in $\mathbb{R}^2 \setminus \{0\}$ for which $\partial\sigma = C_R$.[2]

2 For submission to Davis Lazowski

Problem 2.1. In Problem 2.2 of the previous assignment, you calculated the polar 1–form $\mathrm{d}\theta$ in terms of $\mathrm{d}x$ and $\mathrm{d}y$, where you found

$$\mathrm{d}\theta = \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y.$$

A useful consequence of this is that $\mathrm{d}\theta$ extends to a smooth 1–form on all of $\mathbb{R}^2 \setminus \{0\}$.

1. Use this result to show that there is no way to "correct" the deleted strip $\mathbb{R}^2 \setminus (\mathbb{R}_{\geq 0} \times \{0\})$: show that if f is some other function with $\mathrm{d}f = \mathrm{d}\theta$, then $f = \theta + c$ for some constant c.

[1] This version of the homework dates from April 18, 2017.

[2] Congratulations! You just solved a problem from Math 231a.

2. Let ω be a 1–form on $\mathbb{R}^2 \setminus \{0\}$ with $\mathrm{d}\omega = 0$. Show that there is a constant λ and a function $g\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ with
$$\omega = \lambda\, \mathrm{d}\theta + \mathrm{d}g,$$
i.e., the un-preimage-able 1–form $\mathrm{d}\theta$ is the only defect of the Poincaré Lemma on $\mathbb{R}^2 \setminus \{0\}$.

Problem 2.2. Let c be a singular k–cube, let ω be a k–form, and write
$$c^*\omega = f(x_1, \ldots, x_k)\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k$$
for the pullback. In class, we defined the integral of ω over c by the pullback formula
$$\int_c \omega = \int_{[0,1]^{\times k}} f\, \mathrm{d}x_1 \cdots \mathrm{d}x_k.$$
Let $r\colon [0,1]^{\times k} \to [0,1]^{\times k}$ be a C^∞ bijection with $\det D_x r > 0$ for all x. Show
$$\int_c \omega = \int_{c \circ r} \omega,$$
i.e., the integral of a form is independent of the parametrization of its domain.

Problem 2.3.

1. Let c be a singular 1–cube in $\mathbb{R}^2 \setminus \{0\}$ with $c(0) = c(1)$, and let C_1 be the singular 1–cube parametrizing the unit circle. Show that there is a number n and a 2–chain σ such that $\partial\sigma = nC_1 - c$.[3]
2. Use Stokes's theorem to show that the n associated in this way to c is unique—it does not change even if you choose a different 2–chain σ.

3 For submission to Handong Park

Problem 3.1. For ω a nonzero k–form, show that there is a singular k–cube c with $\int_c \omega \neq 0$. Using $\partial\partial\sigma = 0$, conclude $\mathrm{d}(\mathrm{d}\omega) = 0$ (i.e., mixed partials commute).

Problem 3.2. Let $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a complex polynomial, considered as a function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. Let C_R denote the circle centered at the origin of radius R, and consider the curve $f \circ C_R$.

1. Define a singular 2–cube σ by the formula
$$\sigma(t, x) = t \cdot (f \circ C_R(x)) + (1 - t) \cdot R(\cos(2\pi n x) + i \sin(2\pi n x)).$$
Show that this interpolates between $f \circ C_R$ and nC_R in the sense that $\partial\sigma = f \circ C_R - nC_R$.
2. Show that for $R \gg 0$, σ factors through $\mathbb{R}^2 \setminus \{0\}$.
3. Conclude from Problem 1.2 the fundamental theorem of algebra: the polynomial f has a root in $\mathbb{C}$.

4 For submission to Rohil Prasad

In this section, we continue the analysis of functions on the complex plane initiated above. This is quite long—good luck!

[3]Spivak suggests that you split the domain of c into subintervals with either nonnegative y–values or nonnegative y–values. This is a good idea, but it really requires you to use the definition of a singular 1–cube as a *smooth* function, so beware.

Problem 4.1. A function $f\colon \mathbb{C} \to \mathbb{C}$ is said to be *complex-differentiable* when the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. Note that this is considerably more complicated than being differentiable as a function $\mathbb{R}^2 \to \mathbb{R}^2$, since we *aren't* taking the norm in the denominator, and so we are multiplying two complex numbers together, which has funny effects. We shorthand "f is continuously complex-differentiable on an open set $A \subseteq \mathbb{C}$" to "f is *holomorphic* on A".

1. Show that $f(z) = z$ is holomorphic and that $f(z) = \overline{z}$ is *not*. Show that the sum, product, and inverse (where nonzero) of holomorphic functions are holomorphic.

2. Write an holomorphic function f as $f(x+iy) = u(x+iy) + iv(x+iy)$ for two functions $u, v\colon \mathbb{R}^2 \to \mathbb{R}$. Demonstrate the Cauchy–Riemann equations:[4]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

3. Let $T\colon \mathbb{C} \to \mathbb{C}$ be a $\mathbb{R}$–linear transformation, where $\mathbb{C}$ is considered as a real vector space on the basis $\{1, i\}$, yielding a matrix presentation

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

   Show that T is multiplication by a complex number if and only if $a = d$ and $b = -c$. Compare this with the previous part: a generic holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ has both complex derivative $f'(a)$ at a point as well as a multivariate derivative $D_a f$. How are these related?

4. Extend the standard operations on 1–forms to complex 1–forms $\omega + i\eta$ by the formulas

$$\mathrm{d}(\omega + i\eta) = \mathrm{d}\omega + i\,\mathrm{d}\eta, \qquad \int_c (\omega + i\eta) = \int_c \omega + i \int_c \eta, \qquad \mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y,$$

$$(\omega + i\eta) \wedge (\psi + i\varphi) = (\omega \wedge \psi - \eta \wedge \varphi) + i(\omega \wedge \varphi + \eta \wedge \psi).$$

   Show that $\mathrm{d}(f\,\mathrm{d}z) = 0$ if and only if $f\colon \mathbb{C} \to \mathbb{C}$ satisfies the Cauchy–Riemann equations.

5. (Cauchy Integral Theorem:) If f is holomorphic on A and c is a closed curve with $c = \partial\sigma$ for some 2–chain σ, then $\int_c f\,\mathrm{d}z = 0$.

6. In the example $f(z) = 1/z$, show $f \cdot \mathrm{d}z = i\,\mathrm{d}\theta + \mathrm{d}h$ for some auxiliary function $h\colon \mathbb{C} \setminus \{0\} \to \mathbb{R}$. Use Problem 1.2 to conclude

$$\int_{C_R} f \cdot \mathrm{d}z = 2\pi i n.$$

7. Let f be holomorphic on $\{z : |z| < 1\}$, and define $g(z) = f(z)/z$ which is holomorphic on $\{z : 0 < |z| < 1\}$. For $0 < R_{\text{in}} < R_{\text{out}} < 1$ as in Problem 1.1, conclude

$$\int_{C_{\text{in}}} \frac{f(z)}{z}\,\mathrm{d}z = \int_{C_{\text{out}}} \frac{f(z)}{z}\,\mathrm{d}z.$$

   Finally, take the limit $R_{\text{in}} \to 0$ and conclude the Cauchy Integral Formula:

$$f(0) = \frac{1}{2\pi i} \int_{C_{\text{out}}} \frac{f(z)}{z}\,\mathrm{d}z.$$

   This formula is super remarkable: note that the value of f at 0 is completely determined by its values on the unit circle—all of which are very far away from 0!

[4]Hint: make the approach for $h \to 0$ along the two standard axes.