Homework #7 Solutions

Thayer Anderson, Davis Lazowski, Handong Park, Rohil Prasad
Eric Peterson

1 For submission to Thayer Anderson

Problem 1.1. Let $R_{\mathrm{out}} > R_{\mathrm{in}} > 0$ be the positive radii of two concentric circles in $\mathbb{R}^2$, both centered at the origin. Call the circles C_{out} and C_{in}. Construct a 2–chain σ with $\partial\sigma = C_{\mathrm{out}} - C_{\mathrm{in}}$.

Solution. Here it is:

$$\sigma(r, s) = (r \cdot R_{\mathrm{in}} + (1 - r) \cdot R_{\mathrm{out}}) \cdot \begin{pmatrix} \cos 2\pi s \\ \sin 2\pi s \end{pmatrix}.$$

The 1–chain $\partial\sigma$ has the following four edges:

- Fix $r = 0$: this gives $\sigma(0, s) = R_{\mathrm{out}} \cdot \begin{pmatrix} \cos 2\pi s \\ \sin 2\pi s \end{pmatrix} = C_{\mathrm{out}}$.
- Fix $r = 1$: this gives $\sigma(1, s) = R_{\mathrm{in}} \cdot \begin{pmatrix} \cos 2\pi s \\ \sin 2\pi s \end{pmatrix} = C_{\mathrm{in}}$.
- Fix $s = 0$: this gives $\sigma(r, 0) = (r \cdot R_{\mathrm{in}} + (1 - r) \cdot R_{\mathrm{out}}) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
- Fix $s = 1$: this *also* gives $\sigma(r, 0) = (r \cdot R_{\mathrm{in}} + (1 - r) \cdot R_{\mathrm{out}}) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

In all, this indeed gives $\partial\sigma = C_{\mathrm{out}} - C_{\mathrm{in}}$. (ECP)

Problem 1.2. Again fixing a radius R, let C_R be the circle of radius R centered at the origin.

1. Show $\int_{C_R} \mathrm{d}\theta = 2\pi$, independent of R.
2. Conclude that there is no 2–chain σ in $\mathbb{R}^2 \setminus \{0\}$ for which $\partial\sigma = C_R$.

Solution. 1. This is a matter of doing the calculation. Parametrizing C_R by

$$C_R(t) = R \cdot \begin{pmatrix} \cos 2\pi s \\ \sin 2\pi s \end{pmatrix},$$

we get

$$\begin{aligned} \int_{C_R} \mathrm{d}\theta &= \int_0^1 \left(\left(\frac{x}{x^2 + y^2} \mathrm{d}y - \frac{y}{x^2 + y^2} \mathrm{d}x \right) \Bigg|_{\begin{pmatrix} \cos 2\pi s \\ \sin 2\pi s \end{pmatrix}} \begin{pmatrix} -2\pi \sin 2\pi s \\ 2\pi \cos 2\pi s \end{pmatrix} \right) \mathrm{d}s \\ &= \int_0^1 \left(\frac{\cos 2\pi s}{\cos^2 2\pi s + \sin^2 2\pi s} (2\pi \cos 2\pi s) - \frac{\sin 2\pi s}{\cos^2 2\pi s + \sin^2 2\pi s} (-2\pi \sin 2\pi s) \right) \mathrm{d}s \\ &= \int_0^1 2\pi \, \mathrm{d}s = 2\pi. \end{aligned}$$

2. If this were possible for some fixed choice of R, then Stokes's theorem would give

$$2\pi = \int_{C_R} \mathrm{d}\theta = \int_{\partial\sigma} \mathrm{d}\theta = \int_{\sigma} \mathrm{d}\,\mathrm{d}\theta.$$

On the other hand,

$$\mathrm{d}\left(\frac{x}{\sqrt{x^2+y^2}} \mathrm{d}y - \frac{y}{\sqrt{x^2+y^2}} \mathrm{d}x \right) = 0. \tag{ECP}$$

2 For submission to Davis Lazowski

Problem 2.1. In Problem 2.2 of the previous assignment, you calculated the polar 1–form $\mathrm{d}\theta$ in terms of $\mathrm{d}x$ and $\mathrm{d}y$, where you found

$$\mathrm{d}\theta = \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y.$$

A useful consequence of this is that $\mathrm{d}\theta$ extends to a smooth 1–form on all of $\mathbb{R}^2 \setminus \{0\}$.

1. Use this result to show that there is no way to "correct" the deleted strip $\mathbb{R}^2 \setminus (\mathbb{R}_{\geq 0} \times \{0\})$: show that if f is some other function with $\mathrm{d}f = \mathrm{d}\theta$, then $f = \theta + c$ for some constant c.

2. Let ω be a 1–form on $\mathbb{R}^2 \setminus \{0\}$ with $\mathrm{d}\omega = 0$. Show that there is a constant λ and a function $g\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ with

$$\omega = \lambda\,\mathrm{d}\theta + \mathrm{d}g,$$

i.e., the un-preimage-able 1–form $\mathrm{d}\theta$ is the only defect of the Poincaré Lemma on $\mathbb{R}^2 \setminus \{0\}$.

Solution. 1. Then $df - d\theta = 0$ on the defined region. So $d(f - \theta) = 0$ on this region. Therefore, $f - \theta$ is constant on this region. Now $f = \theta + c$ on this region.

2. Denote the filled-in rectangle of side lengths a, b around the origin by $S_{a,b}$. I claim if and only if $\int_{\partial S_{a,b}} \nu = 0$, then ν is exact. If ν is exact, apply Stokes' theorem.

If $\int_{\partial S_{a,b}} \nu = 0$, then also $\int_{\partial S_{a',b'}} \nu = 0$. To show this,

$$\int_{\partial S_{a,b}} \nu - \int_{\partial S_{a',b'}} \nu = \int_{\partial (S_{a',b'} \Delta S_{a,b})} \nu$$

Where the symmetric difference is a union of closed regions not including the origin, so apply Stokes' theorem again.

Finally, choose a point p_0. For any point p not the origin, there exists a rectangle for which p_0, p are both on the boundary. The size of this rectangle does not matter, as shown above, and since $\int_{\partial S_{a,b}} \nu = 0$, also we know that if we integrate from p to p_0 following one route along the rectangle we'll get the same result as integrating from p to p_0 along the other route: basically, we have proven that the integral from p to p_0 is 'path-independent' along a rectangular path. So choose some rectangular path $P(p, p_0)$ to integrate along. Let

$$g(p) = \int_{P(p_0,p)} \nu$$

Quick differentiation verifies $dg = \nu$. Therefore our claim is proved.

Now let

$$\lambda := \frac{1}{2\pi} \int_{\partial S_{a,b}} \omega$$

Clearly $\int_{C_r} \omega = \int_{\partial S_{a,b}} \omega$ and $\int_{C_r} d\theta = \int_{\partial S_{a,b}} d\theta$, by the exact same application of Stokes' theorem as above. So with this definition of λ, then

$$\int_{\partial S_{a,b}} \omega - \lambda d\theta = 0$$

Now apply our lemma with $\omega - \lambda d\theta = \nu$. (DL)

Problem 2.2. Let c be a singular k–cube, let ω be a k–form, and write

$$c^*\omega = f(x_1, \ldots, x_k)\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k$$

for the pullback. In class, we defined the integral of ω over c by the pullback formula

$$\int_c \omega = \int_{[0,1]^{\times k}} f\, \mathrm{d}x_1 \cdots \mathrm{d}x_k.$$

Let $r\colon [0,1]^{\times k} \to [0,1]^{\times k}$ be a C^∞ bijection with $\det D_x r > 0$ for all x. Show

$$\int_c \omega = \int_{c \circ r} \omega,$$

i.e., the integral of a form is independent of the parametrization of its domain.

Solution. First of all,

$$[c \circ r]^*\omega = r^* c^* \omega = r^*[c^*\omega] = r^*[f dx_1 \wedge \cdots \wedge dx_k] = (f \circ r)(\det r') dx_1 \wedge \cdots \wedge dx_k$$

Therefore,

$$\int_{c \circ r} \omega = \int_{[0,1]^{\times k}} (f \circ r)(\det r') dx_1 \ldots dx_k$$

This is just $\int_{[0,1]^{\times k}} f dx_1 \ldots dx_k$ under the substitution $\vec{x} \to r(\vec{x}')$, since r is orientation-preserving, so substitute back to get the required equality. (DL)

Problem 2.3. 1. Let c be a singular 1–cube in $\mathbb{R}^2 \setminus \{0\}$ with $c(0) = c(1)$, and let C_1 be the singular 1–cube parametrizing the unit circle. Show that there is a number n and a 2–chain σ such that $\partial\sigma = nC_1 - c$.[1]

2. Use Stokes's theorem to show that the n associated in this way to c is unique—it does not change even if you choose a different 2–chain σ.

Solution. 1. We know in polar coordinates we can express $c(t) = (r_c(t), \theta_c(t)$. Therefore $c^* d\theta = \frac{\partial \theta_c}{\partial t} dt$. Therefore

$$\theta_c(t) = \int_0^t c^* d\theta$$

Where we have rotated, without loss of generality, so $\theta(0) = 0$.

Now by problem 1.2, observe that $\int_c d\theta$ therefore is an integral multiple of 2π, where n is the *winding number* around 0 of how many times it goes around 0. Here we implicitly use Stokes' theorem.

Here is the intuitive picture we're using: each time c winds around 0, we need another circle C to cancel out the 'hole' at 0.

Now reduce to the case where $n = 1$, and c winds around 0 precisely once.

[1] Spivak suggests that you split the domain of c into subintervals with either nonnegative y–values or nonnegative y–values. This is a good idea, but it really requires you to use the definition of a singular 1–cube as a *smooth* function, so beware.

In this case, we need to find $\partial\sigma = C - c$. Intuitively, we know what σ needs to 'look like': we have a circle C and a deformed circle c, and σ is just the region between them.

This motivates us to consider a deformation $\mathcal{C}$ between the two circles.

Now let $\mathcal{C}(x,t) := [0,1] \times [0,1] \to \mathbb{R}^2 \setminus \{0\}$ be defined by

$$\mathcal{C}(x,t) = (1-t)c(x) + tC(x)$$

So that $\mathcal{C}(x,0) = c(x)$ and $\mathcal{C}(x,1) = C(x)$.

So

$$\begin{aligned}\partial\mathcal{C} = (\mathcal{C}_{2,0} - \mathcal{C}_{1,0}) - (\mathcal{C}_{2,1} - \mathcal{C}_{1,1}) &= (\mathcal{C}(0,x) - \mathcal{C}(x,0)) - (\mathcal{C}(1,x) - \mathcal{C}(x,1)) \\ &= [C - c] + [\mathcal{C}(0,x) - \mathcal{C}(1,x)] = C - c\end{aligned}$$

This intuition lets us generalise to the general case. This 'straight-line' movement between C and c can only be problematic if it moves between two points $C(x)$ and $c(x)$ on a line including zero. To prevent this, we align the θ components of c and C so that the straight-line movement is just scaling in r.

Break $[0,1]$ into n intervals $[t_i, t_{i+1}]$ by letting $t_0 = 0$, again without loss of generality assuming $c_\theta(t_0) = 0$, and recursively defining t_{i+1} as the unique least point with the following properties:

(a) $t_{i+1} > t_0$

(b) $c_\theta(t_{i+1}) = 0$

(c) If $\mathfrak{t} > t_{i+1}$ and $c_\theta(\mathfrak{t}) = 0$, then $c_\theta[t_{i+1}, \mathfrak{t}) = [0, 2\pi)$.

It's easy to verify that this is the same n as the winding number gives us; for instance, by part b).

We'll compose several deformations, which we could write as one two-chain. First, scale $[t_i, t_{i+1}]$ to $[0,1]$.

Next, deform C_θ to c_θ. C_θ is the identity plus scaling, so we might as well consider $c_\theta(\frac{t}{2\pi})$ and deform it to the identity. We can do this because we can deform $c_\theta(\frac{t}{2\pi})$ reversibly to a single point by shrinking smoothly, and we can do the same for the identity.

Finally, send

$$\mathfrak{h}_i(x,t) := ((1-t)c_r(x), c_\theta(x)) + (tC_r(x), c_\theta(x))$$

The composition of these maps provides the requisite chain over the sub-interval. Summing them gets the answer desired.

2. Suppose we can find chains associated to n, n', say σ, σ'. Then

$$\int_{\partial\sigma'} d\theta - \int_{\partial\sigma} d\theta = 2\pi(n - n')$$

But apply Stokes' theorem to get that the left hand side must be zero since $dd\theta = 0$ and the symmetric difference of σ', σ does not include zero, therefore $n = n'$. (DL)

3 For submission to Handong Park

Problem 3.1. For ω a nonzero k–form, show that there is a singular k–cube c with $\int_c \omega \neq 0$. Using $\partial\partial\sigma = 0$, conclude $\mathrm{d}(\mathrm{d}\omega) = 0$ (i.e., mixed partials commute).

Solution. Confusingly, the ω in the first and second sentences are not identical in function. We treat them in turn:

1. Expand ω in terms of standard k–forms: $\omega = \sum_J \omega_J \, \mathrm{d}x_{J_1} \wedge \cdots \wedge \mathrm{d}x_{J_k}$. Since ω is nonzero at x, at least one of these summands is nonzero; fix some particular J for which this is the case, let $\varepsilon > 0$ be some radius for which $\|x - y\| < \varepsilon$ implies $\omega_J(y) \neq 0$, and consider a k–cube c centered at x in the J–plane of diagonal radius ε' which is entirely contained in this ε–ball. In fact, by continuity of ω_J it follows that either $\omega_J > 0$ or $\omega_J < 0$ everywhere on c; by replacing ω_J with $-\omega_J$ if necessary, we assume that ω_J is everywhere positive. Then, consider the integral

$$\int_c \omega = \int_{[-\varepsilon', \varepsilon']^{\times k}} \omega_J(c(t_1, \ldots, t_k)) \, \mathrm{d}t_1 \cdots \mathrm{d}t_k.$$

   This is a standard integral of a positive function over a positive volume domain, so positive.

2. Let ξ be any $(k-2)$–form, and let $\omega = \mathrm{d}\,\mathrm{d}\xi$ be its double derivative. Suppose that $\omega \neq 0$, and run the argument above to produce a singular k–cube c on which $\int_c \omega \neq 0$. Applying Stokes's theorem twice, we have

$$\int_c \omega = \int_c \mathrm{d}\,\mathrm{d}\xi = \int_{\partial c} \mathrm{d}\xi = \int_{\partial\partial c} \xi = \int_{\varnothing} \xi = 0.$$

   This contradicts the first part, so it must have been the case that $\omega = 0$ all along. (ECP)

Problem 3.2. Let $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a complex polynomial, considered as a function $f \colon \mathbb{R}^2 \to \mathbb{R}^2$. Let C_R denote the circle centered at the origin of radius R, and consider the curve $f \circ C_R$.

1. Define a singular 2–cube σ by the formula

$$\sigma(t, x) = t \cdot (f \circ C_R(x)) + (1 - t) \cdot R(\cos(2\pi n x) + i \sin(2\pi n x)).$$

   Show that this interpolates between $f \circ C_R$ and nC_R in the sense that $\partial\sigma = f \circ C_R - nC_R$.

2. Show that for $R \gg 0$, σ factors through $\mathbb{R}^2 \setminus \{0\}$.

3. Conclude from Problem 1.2 the fundamental theorem of algebra: the polynomial f has a root in $\mathbb{C}$.

Solution.

1. This is the same construction used in Problem 1.1, and it works for the same reason.

2. Last semester we showed that nonzero polynomials are nonzero as functions, and we can reuse that trick here. We impose the following constraint:

$$R > |a_0| + |a_1| + \cdots + |a_{n-1}| + 1.$$

   This feeds into a triangle inequality application as follows:

$$|a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}| \le (|a_0| + |a_1| + \cdots + |a_{n-1}|)|z|^{n-1} < |z|^n.$$

   Applying the triangle inequality again shows that the sum describing the polynomial is always nonzero past this radius—in fact, $f(z)$ is always on the same side of the origin as z^n. It follows directly that the linear interpolator σ never vanishes.

3. For technical reasons that we will fix later, suppose that $f(0) \neq 0$ is nonvanishing at zero. Then for a very *small* radius $r \ll 1$, we have that $\int_{f \circ C_r} \mathrm{d}\theta \ll 1$ is a very small value. Then, we apply Stokes's theorem twice to our supposed 2–chains: the 2–chain $\sigma_{1.1}$ from Problem 1.1 composes with f to give

$$\int_{f \circ C_R} \mathrm{d}\theta = \int_{f \circ C_r} \mathrm{d}\theta + \int_{f \circ \sigma_{1.1}} \mathrm{d}(\mathrm{d}\theta).$$

   Importantly, this *only* works if $f \circ \sigma_{1.1}$ factors through $\mathbb{R}^2 \setminus \{0\}$. Then, we apply Stokes's theorem to the σ from the previous part:

$$\int_{f \circ C_R} \mathrm{d}\theta = \int_{nC_R} \mathrm{d}\theta + \int_{\sigma} \mathrm{d}\,\mathrm{d}\theta + \int_{f \circ \sigma_{1.1}} \mathrm{d}(\mathrm{d}\theta).$$

We previously calculated $\int_{nC_R} \mathrm{d}\theta = 2\pi n$, and the other two integrals are taken of positive functions, so the sum can only be larger. For $n \geq 1$, this sum is not near to zero—a contradiction, indicating that $f \circ \sigma_{1.1}$ does *not* factor through $\mathbb{R}^2 \setminus \{0\}$, which in turn entails the existence of some zero in the range of f. (ECP)

4 For submission to Rohil Prasad

Problem 4.1. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *complex-differentiable* when the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. Note that this is considerably more complicated than being differentiable as a function $\mathbb{R}^2 \to \mathbb{R}^2$, since we *aren't* taking the norm in the denominator, and so we are multiplying two complex numbers together, which has funny effects. We shorthand "f is continuously complex-differentiable on an open set $A \subseteq \mathbb{C}$" to "f is *holomorphic* on A".

1. Show that $f(z) = z$ is holomorphic and that $f(z) = \overline{z}$ is *not*. Show that the sum, product, and inverse (where nonzero) of holomorphic functions are holomorphic.

2. Write a holomorphic function f as $f(x + iy) = u(x + iy) + iv(x + iy)$ for two functions $u, v : \mathbb{R}^2 \to \mathbb{R}$. Demonstrate the Cauchy-Riemann-equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

3. Let $T : \mathbb{C} \to \mathbb{C}$ be a $\mathbb{R}$-linear transformation where $\mathbb{C}$ is considered as a real vector space on the basis $\{1, i\}$, yielding a matrix presentation

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

   Show that T is multiplication by a complex number if and only if $a = d$ and $b = -c$. Compare this with the previous part: a generic holomorphic function $f : \mathbb{C} \to \mathbb{C}$ has both complex derivative $f'(a)$ at a point as well as a multivariate derivative $D_a f$. How are these related?

4. Extend the standard operations on 1-forms to complex 1-forms $\omega + i\eta$ by the formulas

$$d(\omega + i\eta) = d\omega + id\eta, \int_c (\omega + i\eta) = \int_c \omega + i \int_c \eta, dz = dx + idy,$$

   and

$$(\omega + i\eta) \wedge (\psi + i\varphi) = (\omega \wedge \psi - \eta \wedge \varphi) + i(\omega \wedge \varphi + \eta \wedge \psi).$$

   Show that $d(f dz) = 0$ if and only if $f : \mathbb{C} \to \mathbb{C}$ satisfies the Cauchy-Riemann equations.

5. (Cauchy Integral Theorem:) If f is holomorphic on A and c is a closed curve with $c = \partial\sigma$ for some 2-chain σ, then $\int_c f dz = 0$.

6. In the example $f(z) = 1/z$, show $f \cdot dz = id\theta + dh$ for some auxiliary function $h : \mathbb{C}$ $\{0\} \to \mathbb{R}$. Use Problem 1.2 to conclude

$$\int_{C_R} f \cdot dz = 2\pi i n.$$

7. Let f be holomorphic on $\{z : |z| < 1\}$, and define $g(z) = f(z)/z$ which is holomorphic on $\{z : 0 < |z| < 1\}$. For $0 < R_{in} < R_{out} < 1$ as in Problem 1.1, conclude

$$\int_{C_{in}} \frac{f(z)}{z} dz = \int_{C_{out}} \frac{f(z)}{z} dz.$$

Finally, take the limit $R_{in} \to 0$ and conclude the Cauchy Integral Formula:

$$f(0) = \frac{1}{2\pi i} \int_{C_{out}} \frac{f(z)}{z} dz.$$

This formula is super remarkable: note that the value of f at 0 is completely determined by its values on the unit circle–all of which are very far away from 0!

Solution. 1. If $f(z) = z$, then

$$\lim_{h\to 0}(f(z+h) - f(z))/h = \lim_{h\to 0}(z+h-z)/h = 1.$$

Therefore, $f(z) = z$ is holomorphic.

On the other hand, take $f(z) = \overline{z}$.

Let t be a real parameter going to 0. Then $\lim_{t\to 0}(f(z+t) - f(z))/t = \lim_{t\to 0}(\overline{z}+t-\overline{z})/t = 1$. However, if we take the limit in the imaginary direction, we get $\lim_{t\to 0}(f(z+it) - f(z))/it = \lim_{t\to 0}(\overline{z}-it-\overline{z})/t = -1$. Since the limit is different in two directions, the limit does not exist on the complex plane and $f(z) = \overline{z}$ is not holomorphic.

The fact that the sum of two holomorphic functions is holomorphic follows by the linearity of the complex derivative as defined above.

Now take two holomorphic functions f, g. Then

$$\begin{aligned}
\lim_{h\to 0} \frac{fg(z+h) - fg(z)}{h} &= \lim_{h\to 0} \frac{f(z+h)g(z+h) - f(z)g(z+h) + f(z)g(z+h) - f(z)g(z)}{h} \\
&= \lim_{h\to 0} \frac{g(z+h)(f(z+h) - f(z)) + f(z)(g(z+h) - g(z))}{h} \\
&= \lim_{h\to 0} g(z+h) \cdot \frac{f(z+h) - f(z)}{h} + \lim_{h\to 0} f(z) \cdot \frac{g(z+h) - g(z)}{h} \\
&= g(z)f'(z) + f(z)g'(z).
\end{aligned}$$

Therefore, fg is holomorphic as well.

The inverse is holomorphic by a similar calculation:

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{1}{f(z+h)} - \frac{1}{f(z)}}{h} &= \lim_{h\to 0} \frac{f(z) - f(z+h)}{f(z)f(z+h)h} \\
&= \frac{1}{f(z)^2} \lim_{h\to 0} \frac{f(z) - f(z+h)}{h} \\
&= -\frac{f'(z)}{f(z)^2}.
\end{aligned}$$

2. For a real parameter t going to 0, we have by definition

$$\lim_{t\to 0} \frac{f(z+t) - f(z)}{t} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and

$$\lim_{t\to 0} \frac{f(z+it) - f(z)}{it} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Since f is holomorphic, both of these limits coincide. Equating the real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

3. Multiplication by a constant number $x+yi$ sends 1 to $x+yi$ and i to $-y+xi$. Therefore, the associated matrix is

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

From this, we can conclude that T is multiplication by a complex number iff $a = d$ and $b = -c$. Furthermore, from our derivation we find that in this case T is multiplication by the complex number $a - bi$.

By the Cauchy-Riemann equations above, we find that the Jacobian $D_a f$ satisfies exactly this relation, and so as a transformation is equivalent to multiplication by the complex number $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$. This is exactly $f'(a)$ by our derivation above!

4. Write out $f = u + iv$, $dz = dx + idy$. Then, we get $fdz = (udx - vdy) + i(vdx + udy)$.

Taking the exterior derivative, we calculate

$$\begin{aligned} d(fdz) &= (\frac{\partial u}{\partial y}dy \wedge dx - \frac{\partial v}{\partial x}dx \wedge dy) + i(\frac{\partial v}{\partial y}dy \wedge dx + \frac{\partial u}{\partial x}dx \wedge dy) \\ &= [(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}) + i(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})]dx \wedge dy. \end{aligned}$$

Therefore, $d(fdz)$ is equal to 0 iff the Cauchy-Riemann equations are satisfied.

5. We apply Stokes' Theorem:

$$\int_c fdz = \int_\sigma d(fdz) = \int_\sigma 0 = 0.$$

6. We can write $f(x+iy) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$. Therefore, $f = u + iv$ where $u(x+iy) = \frac{x}{x^2+y^2}$, $v(x+iy) = -\frac{y}{x^2+y^2}$.

By our formula for $d\theta$, we get $d\theta = vdx + udy$. Therefore, it follows that $fdz = id\theta + (udx - vdy)$, so we must find some h such that $dh = udx - vdy$. This is equivalent to showing that there exists some function h such that $\frac{\partial h}{\partial x} = u$, $\frac{\partial h}{\partial y} = -v$.

Luckily for us, we can construct this function explicitly as $h(x+iy) = \frac{1}{2}\log(x^2+y^2)$.

We can then evaluate

$$\int_{C_R} f \cdot dz = i \cdot \int_{C_R} d\theta + \int_{C_R} dh$$

.

By Problem 1.2, $i \cdot \int_{C_R} d\theta = 2\pi i$. On the other hand, by Stokes' Theorem, $\int_{C_R} dh = \int_\emptyset h = 0$. Therefore, we get

$$\int_{C_R} f \cdot dz = 2\pi i.$$

7. As in Problem 1.1, let σ be a 2-chain with $\partial\sigma = C_{out} - C_{in}$.

Then by Stokes' theorem and the fact that g is holomorphic,

$$\int_{C_{out}} g(z)dz - \int_{C_{in}} g(z)dz = \int_\sigma d(g(z)dz) = 0.$$

As we take R_{in} to 0, we have $f(z)$ approaches $f(0)$. Therefore,

$$\lim_{R_{in}\to 0} \int_{C_{in}} \frac{f(z)}{z}dz = \lim_{R_{in}\to 0} \int_{C_{in}} \frac{f(0)}{z}dz.$$

By the previous part, the RHS is equal to $2\pi i \cdot f(0)$. Then, by the equality we have already established we conclude

$$f(0) = \frac{1}{2\pi i} \int_{C_{out}} \frac{f(z)}{z} dz. \tag{RP}$$