Homework #6

Math 25b

Due: April 12th, 2017

Guidelines:

- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one, a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!)[1]

1 For submission to Thayer Anderson

Problem 1.1. Akin to the tensor product of functionals described in class, you can also define a tensor product of vectors: $V \otimes W$ is the vector space populated by formal sums of elements of the form $v \otimes w$, subject to the relations

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \qquad (kv) \otimes w = k(v \otimes w) = v \otimes (kw),$$
$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2.$$

(The tensor described in class is thus this definition of tensor, applied to the dual space $\mathbb{R}^*$.) Similarly, you can also build a wedge product of vectors $v_1, \ldots, v_k \in V$ as a particular kind of tensor:

$$v_1 \wedge \cdots \wedge v_k = \frac{1}{k!} \sum_{\sigma \text{ a permutation of } \{1,\ldots,k\}} \operatorname{sign}(\sigma) \cdot (v_{\sigma 1} \otimes \cdots \otimes v_{\sigma k}),$$

considered as a vector in $(\mathbb{R}^n)^{\otimes k}$. (Again, the wedge product defined in class is thus this definition of wedge product, applied to the dual space $\mathbb{R}^*$.)

1. Show that a linear map $V \otimes W \to U$ is identical information to a bilinear function $V \times W \to U$.
2. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$, and let $\varphi_1, \ldots, \varphi_n$ be the dual basis under the standard inner product. Demonstrate the identity

   $$(\varphi_{j_1} \wedge \cdots \wedge \varphi_{j_k})(e_{j_1} \wedge \cdots \wedge e_{j_k}) = 1$$

   by considering the input as a vector in $(\mathbb{R}^n)^{\otimes k}$ and the function as a vector in $((\mathbb{R}^n)^*)^{\otimes k} = ((\mathbb{R}^n)^{\otimes k})^*$.

[1] This version of the homework dates from April 11, 2017.

3. Remark on the role of the binomial/factorial coefficient in the definition of the wedge product. If that factor were omitted, what would the above pairing evaluate to instead? Why?

4. Note that if $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_\ell\}$ are basis of V and W respectively, then $\{v_i \otimes w_j\}$ forms a basis of $V \otimes W$. Conclude more generally that if $v_1, \ldots, v_k$ is a k–tuple of vectors in $\mathbb{R}^n$ and $\psi_1, \ldots, \psi_k$ is a k–tuple of linear functionals on $\mathbb{R}^n$, then the following two values agree:

$$(\psi_1 \wedge \cdots \wedge \psi_k)(v_1, \ldots, v_k) = (\psi_1 \wedge \cdots \wedge \psi_k)(v_1 \wedge \cdots \wedge v_k).$$

Problem 1.2. For $f, g\colon \mathbb{R}^n \to \mathbb{R}$, demonstrate the *Leibniz rule*

$$d(f \cdot g) = df \cdot g + f \cdot dg.$$

2 For submission to Davis Lazowski

Problem 2.1. Recall the definition of the cross-product in $\mathbb{R}^3$: for $v, w \in \mathbb{R}^3$, $v \times w$ is the vector representing the linear functional

$$\varphi(u) = \det \begin{pmatrix} u \mid v \mid w \end{pmatrix}$$

under the standard inner product—i.e., $\varphi(u) = \langle u, v \times w \rangle$. Demonstrate the following truckload of identities:

1.

$$\begin{aligned} e_1 \times e_1 &= 0, & e_1 \times e_2 &= e_3, & e_1 \times e_3 &= -e_2, \\ e_2 \times e_1 &= -e_3, & e_2 \times e_2 &= 0, & e_2 \times e_3 &= e_1, \\ e_3 \times e_1 &= e_2, & e_3 \times e_2 &= -e_1, & e_3 \times e_3 &= 0. \end{aligned}$$

2. $v \times w = (v_2 w_2 - v_3 w_2)e_1 + (v_3 w_1 - v_1 w_3)e_2 + (v_1 w_2 - v_2 w_1)e_3$.

3. $\|v \times w\| = \|v\| \cdot \|w\| \cdot \sin\theta$, where θ is the angle formed by v and w as rays intersecting at the origin. Conclude $\langle v \times w, v \rangle = 0$ and $\langle v \times w, w \rangle = 0$.

4. The juggling identities: $\langle v, w \times u \rangle = \langle w, u \times v \rangle = \langle u, v \times w \rangle$.

5. The associative identities: $v \times (w \times u) = \langle v, u \rangle w - \langle v, w \rangle u$ and $(v \times w) \times u) = \langle v, u \rangle w - \langle w, u \rangle v$.

6. $\|v \times w\| = \sqrt{\|v\|^2 \cdot \|w\|^2 - \langle v, w \rangle^2}$.

Problem 2.2. Recall the polar coordinate transformation $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta$, defined for $0 < \theta < 2\pi$ and $r > 0$. Prove that where θ is defined as a function of x and y, we have

$$d\theta = \frac{-y}{x^2 + y^2}\,\mathrm{d}x + \frac{x}{x^2 + y^2}\,\mathrm{d}y.$$

3 For submission to Handong Park

Problem 3.1. For $f\colon \mathbb{R}^n \to \mathbb{R}$, we define a *vector field* ∇f by the formula

$$\nabla f\colon (a \in \mathbb{R}^n) \mapsto \begin{pmatrix} \left.\frac{\partial f}{\partial x_1}\right|_{x=a} \\ \vdots \\ \left.\frac{\partial f}{\partial x_n}\right|_{x=a} \end{pmatrix} \in T_a\mathbb{R}^n.$$

Recall also the directional derivative from Homework #3: given a tangent vector $v \in T_a\mathbb{R}^n$, we set

$$\mathbb{D}_a^v f = \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t}.$$

Conclude $\mathbb{D}_a^v(f) = \langle v, \nabla f(a) \rangle$, and hence that $\nabla f(a)$ is the direction of greatest ascent[2].

[2]Or "direction of fastest change", if you prefer.

Problem 3.2. Let $f\colon U \to \mathbb{R}^n$ be a differentiable function with a differentiable inverse $f^{-1}\colon f(U) \to \mathbb{R}^n$. If every closed form on U is exact, show that the same is true for $f(U)$.

4 For submission to Rohil Prasad

Problem 4.1. Let $c\colon [0,1] \to (\mathbb{R}^n)^n$ be a 1–parameter continuous family of families of n vectors in $\mathbb{R}^n$, and suppose that $c(t) = \{c_1(t), \dots, c_n(t)\}$ is a basis of $\mathbb{R}^n$ for each $0 \le t \le 1$. Show that the *orientation* of each basis must be the same, i.e., the value

$$[c_1(t), \dots, c_n(t)] := \operatorname{sign}(\det(c_1(t) \mid \dots \mid c_n(t)))$$

is constant even as t varies.

Problem 4.2. In class, we "proved" by example that any quadratic form $Q\colon \mathbb{R}^n \to \mathbb{R}$ can be written in the form

$$Q = a_1^2 + \cdots + a_k^2 - b_1^2 - \cdots - b_\ell^2$$

for a family of linearly independent linear functionals a_* and b_*. Complete our discussion by turning our examples into an honest proof. (Don't worry about the invariance of the signature; just work on this existence half.)