Homework #5

Math 25b

Due: March 22nd, 2017

Guidelines:

- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one, a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!)[1]

1 For submission to Thayer Anderson

Problem 1.1. Let P be a partition of $[a, b]$ into subintervals S_j, and for each S_j choose a point $s_j \in S_j$. We will call this data a *marked partition*[2], and we say that the maximum volume of any rectangle in the partition is the *mesh size* of P.

For an integrable function $f\colon [a, b] \to \mathbb{R}$ and any $\varepsilon > 0$, show that there must exist a corresponding $\delta > 0$ such that if the *mesh size* of a marked partition P is less than δ then

$$\sum_{S_j} \left| f(s_j)\operatorname{vol}(S_j) - \int_{S_j} f(x)\,\mathrm{d}x \right| < \varepsilon.$$

(Hint: Have a look at the proof of our characterization of the "integrability" condition for clues.)

Problem 1.2. Let $\{f_n\}$ be a *nondecreasing* sequence of integrable functions defined on an interval $[a, b]$, and suppose that for each $x \in [a, b]$ the limit $\lim_{n\to\infty} f_n(x)$ exists. We use this to define a new function $f(x) = \lim_{n\to\infty} f_n(x)$.

Suppose that this new function is itself Riemann integrable. We will conclude the limit interchange law

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \left(\int_a^b f_n(x)\,\mathrm{d}x \right).$$

[1] This version of the homework dates from March 20, 2017.

[2] In practice, we have been concerned with s_j that mark minima or maxima of $f|_{S_j}$ for some function f.

0. We first need to establish some vocabulary; there's nothing to prove in this bullet point. Consider the sequence of errors $g_n = f - f_n$, and write $\eta = \varepsilon/(b - a + 1)$ for an auxiliary error usefully satisfying $\eta \operatorname{vol}[a, b] < \varepsilon$. For each n, use the previous Problem to select a δ_n such that if the mesh size of a marked partition P is beneath δ_n, then

$$\sum_{S_j \in P} \left| g_n(s_j) \operatorname{vol}(S_j) - \int_{S_j} g_n(x)\,\mathrm{d}x \right| < 2^{-n}\eta.$$

1. For each x, let $N(x)$ denote the first integer such that $n \geq N$ forces $g_n(x) < \eta$, and then let $\delta(x) = \delta_{N(x)}$. Show that there exists a marked partition satisfying $\operatorname{vol}(S_j) < \delta(s_j)$.

2. Write P_n for the collection of those rectangles S_j in the partition (guaranteed by the previous part) which satisfy $N(s_j) = n$; since there are finitely many rectangles in the partition, we let N denote the largest n so appearing. Justify all of the steps in the following sequence:

$$\begin{aligned} 0 \leq \int_a^b g_N(x)\,\mathrm{d}x &= \sum_{S_j \in P} \int_{S_j} g_N(x)\,\mathrm{d}x = \sum_{n=1}^{N} \sum_{S_j \in P_n} \int_{S_j} g_N(x)\,\mathrm{d}x \leq \sum_{n=1}^{N} \sum_{S_j \in P_n} \int_{S_j} g_n(x)\,\mathrm{d}x \\ &\leq \sum_{n=1}^{N} \left(\eta 2^{-n} + \sum_{S_j \in P_n} g_n(s_j) \operatorname{vol}(S_j) \right) < \sum_{n=1}^{N} \left(\eta 2^{-n} + \sum_{S_j \in P_n} \eta \operatorname{vol}(S_j) \right) < \eta(b - a + 1) = \varepsilon. \end{aligned}$$

3. Use this to finish the argument and conclude the interchange law.

Problem 1.3. We now use Problem 1.2 to describe a new kind of integrability for *unbounded* functions. Given an unbounded function $f\colon [a, b] \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, we set its N^{th} truncation to be $f^{\leq N}(x) = \min\{f(x), N\}$. We say that f is *improperly integrable* when the limit

$$\lim_{N \to \infty} \int_a^b f^{\leq N}(x)\,\mathrm{d}x =: \int_a^b f(x)\,\mathrm{d}x$$

exists.

1. If f is integrable to begin with, show that $f^{\leq N}$ is integrable. Conclude also that if f is integrable to begin with, then its improper integral agrees with its integral.

2. Show that if $f\colon [a, b] \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is improperly integrable but not integrable, then the set of points where it takes the value ∞ is measure zero.

3. Show that this problem is compatible with the above one: for a sequence $f_n\colon [a, b] \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ of integrable functions converging pointwise to an integrable function $f\colon [a, b] \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, demonstrate the interchange identity

$$\lim_{n \to \infty} \int f_n(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x,$$

where the integrals are interpreted as improper integrals.

2 For submission to Davis Lazowski

Our goal in this section is to make the computation

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi},$$

which is a formula that comes up often when studying normal distributions (or "bell curves"). We will do this in many parts — I suggest doing them in order.[3]

Problem 2.1. Define $f\colon \{r \mid r > 0\} \times (0, 2\pi) \to \mathbb{R}^2$ by $f(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$. Show that f is injective, compute $D_{(r,\theta)}f$, and show that $\det D_{(r,\theta)}f \neq 0$ for all (r, θ). Show that the image of f is the following set A from an old homework:

$$A = \mathbb{R} \setminus ([0, \infty) \times \{0\}) = \{(x, y) \in \mathbb{R}^2 \mid \text{if } x \geq 0, \text{ then } y \neq 0\}.$$

Problem 2.2. We write $P = f^{-1}$ for the inverse function, so named after the *polar coordinate system.* Show that the components $P(x, y) = (r(x, y), \theta(x, y))$ take the form

$$r(x,y) = \sqrt{x^2 + y^2}, \qquad \theta(x,y) = \begin{cases} \tan^{-1} y/x & \text{if } x > 0,\ y > 0, \\ \pi + \tan^{-1} y/x & \text{if } x < 0, \\ 2\pi + \tan^{-1} y/x & \text{if } x > 0,\ y < 0, \\ \pi/2 & \text{if } x = 0,\ y > 0, \\ 3\pi/2 & \text{if } x = 0,\ y < 0. \end{cases}$$

Calculate $D_{(x,y)}P$.

Problem 2.3.

1. Fix values r_1, r_2, θ_1, and θ_2. Let $C \subseteq A$ be the region between the circles of radii r_1 and r_2 and the half-lines through 0 which make angles of θ_1 and θ_2 with the x–axis. If $h\colon C \to \mathbb{R}$ is integrable and $h(x, y) = g(r(x, y), \theta(x, y))$, show that

$$\int_C h = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} g(r, \theta) r \,\mathrm{d}\theta \,\mathrm{d}r.$$

2. Fix a value ρ, and let $B = \{(x, y) \mid x^2 + y^2 \leq \rho^2\}$. Show that

$$\int_B h = \int_0^{\rho} \int_0^{2\pi} g(r, \theta) r \,\mathrm{d}\theta \,\mathrm{d}r.$$

Problem 2.4. Again fix a value ρ, and set $D = [-\rho, \rho] \times [-\rho, \rho]$. Show both of the equalities

$$\int_B e^{-(x^2+y^2)} \,\mathrm{d}x \,\mathrm{d}y = \pi(1 - e^{-\rho^2}), \qquad \int_D e^{-(x^2+y^2)} \,\mathrm{d}x \,\mathrm{d}y = \left(\int_{-\rho}^{\rho} e^{-z^2} \,\mathrm{d}z \right)^2.$$

(Hint: It may be useful to solve Problem 4.1 first.)

Problem 2.5. Prove the limit equality

$$\lim_{\rho \to \infty} \int_B e^{-(x^2+y^2)} \,\mathrm{d}x \,\mathrm{d}y = \lim_{\rho \to \infty} \int_D e^{-(x^2+y^2)} \,\mathrm{d}x \,\mathrm{d}y,$$

and hence conclude

$$\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}.$$

[3] This is problem 3-41 from Spivak, which concludes with some very unpleasant hero-worship. I advise ignoring Spivak, Lord Kelvin (who he attributes the quote to), and for that matter most of what Hardy says in his famous *Apology.*

3 For submission to Handong Park

Problem 3.1. Let $g_1, g_2 \colon \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable, and as in a homework problem from a few weeks ago assemble these into a function f according to

$$f(x,y) = \int_0^x g_1(t,0)\,\mathrm{d}t + \int_0^y g_2(x,t)\,\mathrm{d}t.$$

Back then we complained about how f can be poorly behaved if $\frac{\partial g_1}{\partial y}$ does not agree with $\frac{\partial g_2}{\partial x}$, so now we instead assume that they are equal. Show that under this assumption we have $\frac{\partial f}{\partial x}(x,y) = g_1(x,y)$.

Problem 3.2. Our goal in this Problem is to show that if U is a rectangle and g is a linear transformation, then the volume of $g(U)$ is the volume of U scaled up by $|\det g|$.

1. Let $g \colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation of one of the following types:
   (a) Fixing an index n and a scalar k, g takes the form $g(e_j) = \begin{cases} ke_n & \text{if } j = n, \\ e_j & \text{otherwise.} \end{cases}$
   (b) Fixing indices m and n, g takes the form $g(e_j) = \begin{cases} e_n + e_m & \text{if } j = n, \\ e_j & \text{otherwise.} \end{cases}$
   (c) Fixing indices m and n, g takes the form $g(e_j) = \begin{cases} e_m & \text{if } j = n, \\ e_n & \text{if } j = m, \\ e_j & \text{otherwise.} \end{cases}$
2. Conclude from this the case of a general linear transformation g.

Problem 3.3. Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function. Use Fubini's theorem to show that if $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are both continuous, then they are equal. (Hint: If their difference is nonzero at any point, then by continuity it is nonzero on an open rectangle containing that point.)

Problem 3.4. Let $f \colon [a,b] \times [c,d] \to \mathbb{R}$ be continuous, and suppose that $\frac{\partial f}{\partial y}$ exists and is continuous. Define $F(y) = \int_a^b f(x,y)\,\mathrm{d}x$. Using the classical fundamental theorem of calculus, prove the following identity, known as *Leibniz's law*:

$$F'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x.$$

That is: differentiation and integration commute. (Hint: $F(y) = \int_a^b f(x,y)\,\mathrm{d}x = \int_a^b (\int_c^y \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}y + f(x,c))\,\mathrm{d}x$.)[4]

4 For submission to Rohil Prasad

Our goal in this section is to make the famous computation

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

which made a very brief appearance at the end of Math 25a as we were discussing Dirichlet's theorem — specifically, the left-hand side we called $\zeta(2)$. We will also do this in many parts, and I again advise doing them in order.

[4] You'll discover that you can get away with much less than continuity of $\partial f / \partial y$.

Problem 4.1. 1. Suppose $f, g\colon [a,b] \times [c,d] \to \mathbb{R}$ are integrable functions on a common domain D, and suppose that f is independent of the second variable and that g is independent of the first. Conclude the formula

$$\int_c^d \int_a^b f \cdot g \,\mathrm{d}x \,\mathrm{d}y = \left(\int_a^b f(x,0) \,\mathrm{d}x \right) \cdot \left(\int_c^d g(0,y) \,\mathrm{d}y \right).$$

2. Use this to calculate

$$\int_0^1 \int_0^1 x^{n-1} y^{n-1} \,\mathrm{d}x \,\mathrm{d}y = \frac{1}{n^2}.$$

Problem 4.2. Sum the above identity over n, then use Problem 1.2 and Problem 1.3 to conclude that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \int_0^1 \int_0^1 \frac{1}{1-xy} \,\mathrm{d}x \,\mathrm{d}y.$$

Problem 4.3. There are several ways to compute this integral.[5]

1. Demonstrate

$$\int_0^1 \int_0^1 \left(\frac{1}{1-xy} - \frac{1}{1+xy} \right) \mathrm{d}x \,\mathrm{d}y = \frac{1}{2}\zeta(2)$$

by using the alternative coordinate system

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}.$$

2. Demonstrate also

$$\int_0^1 \int_0^1 \left(\frac{1}{1-xy} + \frac{1}{1+xy} \right) \mathrm{d}x \,\mathrm{d}y = 2 \int_0^1 \int_0^1 \frac{1}{1-x^2y^2} \,\mathrm{d}x \,\mathrm{d}y.$$

(Note the sign change.)

3. Conclude

$$\zeta(2) = \frac{4}{3} \int_0^1 \int_0^1 \frac{1}{1-x^2y^2} \,\mathrm{d}x \,\mathrm{d}y.$$

Problem 4.4. Finally, use the second coordinate change

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin r / \cos s \\ \sin s / \cos r \end{pmatrix}$$

to compute the last integral in the preceding step.

1. Calculate the image of the unit square in xy–coordinates through the coordinate transform to be the triangle with vertices at $(0,0)$, $(\pi/2, 0)$, and $(0, \pi/2)$ in rs–coordinates.

2. Calculate the Jacobian of the transformation.

3. Use the change of coordinates theorem and evaluate the resulting integral.

At this point you're probably sick of this, but this method of proof generalizes to the study of "polyzeta functions", which have some real importance in number theory. You could look them up for some extra fun.

[5] A previous version of this assignment had a different one, which you can find as "Proof #3" in `http://math.cmu.edu/~bwsulliv/MathGradTalkZeta2.pdf`. That one requires a clever substitution but is otherwise straightforward. The substitution in this one is easier but requires some inspired intermediate algebraic steps.