Homework #4 Solutions

Thayer Anderson, Davis Lazowski, Handong Park, Rohil Prasad
Eric Peterson

Throughout, $A \subseteq \mathbb{R}^n$ will be a compact rectangle.[1]

1 For submission to Thayer Anderson

Problem 1.1. This problem shows that it is possible, if tedious, to perform integration of simple functions straight from the definitions. Define $f\colon [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x + y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Without invoking Fubini's theorem, show that f is integrable and that $\int_A f = 1/2$.

Solution. We manually construct a cofinal sequence of mutually refining partitions and calculate their lower and upper Darboux sums. Let P_k denote the partition with cut points $j2^{-k}$ on both axes for $0 < j < 2^k$. Consider just P_1, which subdivides the square $[0,1]^{\times 2}$ into four quadrants, where the maximum and minimum values of f are easily calculated:

$$\begin{aligned} \mathcal{L}_f(P_1) &= 1 \cdot 1/4 + 0 \cdot 1/4 + 0 \cdot 1/4 + 0 \cdot 1/4 = 1/4, \\ \mathcal{U}_f(P_1) &= 1 \cdot 1/4 + 1 \cdot 1/4 + 1 \cdot 1/4 + 0 \cdot 1/4 = 3/4. \end{aligned}$$

In fact, this calculation is generic enough to give a recursive formula. Passing from P_1 to P_2, for instance, causes no changes in the upper- and lower-sums calculated over the quadrants where f is identically 1 or 0, but in the two half-filled quadrants it replaces $1/4 \cdot 1$ with $1/4 \cdot \mathcal{U}_f(P_1)$ in the upper sum and $1/4 \cdot 0$ with $1/4 \cdot \mathcal{L}_f(P_1)$ in the lower sum. This pattern continues:

$$\mathcal{L}_f(P_{j+1}) = \mathcal{L}_f(P_j) + 2^j \cdot \left(\frac{1}{2^{j+1}}\right)^2, \qquad \mathcal{U}_f(P_{j+1}) = \mathcal{U}_f(P_j) - 2^j \cdot \left(\frac{1}{2^{j+1}}\right)^2,$$

which also admit the nonrecursive expressions

$$\mathcal{L}_f(P_{j+1}) = \sum_{j=1}^{\infty} 2^{-(j+1)} = 1/2, \qquad \mathcal{U}_f(P_{j+1}) = 1 \sum_{j=1}^{\infty} 2^{-(j+1)} = 1/2.$$

Since the collections of upper and lower sums thus have the same respective infimum and supremum, f is integrable with integral 1/2. (ECP)

Problem 1.2.

1. Let $f, g\colon A \to \mathbb{R}$ be integrable functions which satisfy $f \le g$. Show that this transfers to their integrals: $\int_A f \le \int_A g$.
2. Let $f\colon A \to \mathbb{R}$ be integrable. Show that $|f|$ is also integrable, and conclude $\left|\int_A f\right| \le \int_A |f|$.

[1] You might also like to be reminded that integrable functions are defined to be bounded.

Solution. 1. The inequality $f \leq g$ transfers to infima and suprema over a subrectangle S:

$$\sup\{f(x) \mid x \in S\} \leq \sup\{g(x) \mid x \in S\}, \qquad \inf\{f(x) \mid x \in S\} \leq \inf\{g(x) \mid x \in S\}.$$

Since, e.g., the first expression is also written $M_S(f)$, by summing over S we learn that the inequality also transfers to upper and lower Darboux sums:

$$\mathcal{U}_P(f) \leq \mathcal{U}_P(g), \qquad \mathcal{L}_P(f) \leq \mathcal{L}_P(g).$$

The "sandwich theorem" for limits finishes the proof.

2. Since f is integrable, it is bounded, so we consider it as a function $f\colon A \to [m, M]$. Let $g\colon [m, M] \to \mathbb{R}$ be any continuous function; then $g \circ f$ is integrable, since bounded functions are integrable if and only if their sets of positive oscillation are measure zero, which is true here. Since $g(x) = |x|$ is such a function, $|f|$ is integrable.

Alternatively, if f and g are integrable functions, then $M(x) = \max\{f(x), g(x)\}$ and $m(x) = \min\{f(x), g(x)\}$ are both integrable, which entails that $f_{\geq 0} = \max\{f(x), 0\}$ and $f_{\leq 0} = \min\{f(x), 0\}$ are integrable. This gives a decomposition

$$\int_A |f| = \int_A |f_{\geq 0} + f_{\leq 0}| = \int_A |f_{\geq 0}| + \int_A |f_{\leq 0}| \geq \int_A f_{\geq 0} + \int_A f_{\leq 0} = \int_A (f_{\geq 0} + f_{\leq 0}) = \int_A f. \qquad \text{(ECP)}$$

Problem 1.3. If $f\colon A \to \mathbb{R}$ is nonnegative and integrable with $\int_A f = 0$, show that any any $\varepsilon > 0$ the set $\{x \in A \mid f(x) > \varepsilon\}$ has measure zero.

Solution. We prove the contrapositive: we suppose that for some $\varepsilon > 0$ the set $B = \{x \in A \mid f(x) > \varepsilon\}$ does not have measure zero and we set out to show that $\int_A f$ is nonzero. In particular, there must be some $\eta > 0$ for which any cover of B by rectangles has total volume at least η. Let P be any partition; then we split the upper sum into two pieces:

$$\mathcal{U}_P(f) = \sum_{S \in P} M_f(S) \cdot \mathrm{vol}(S) = \sum_{\substack{S \in P \\ S \cap B \neq \emptyset}} M_f(S) \cdot \mathrm{vol}(S) + \sum_{\substack{S \in P \\ S \cap B = \emptyset}} M_f(S) \cdot \mathrm{vol}(S).$$

Since all the summands are positive, this total sum is bounded below by just the first sum, which in turn is bounded below by a constant:

$$\sum_{\substack{S \in P \\ S \cap B \neq \emptyset}} M_f(S) \cdot \mathrm{vol}(S) \geq \sum_{\substack{S \in P \\ S \cap B \neq \emptyset}} \varepsilon \cdot \mathrm{vol}(S) \geq \varepsilon \cdot \eta > 0.$$

There is therefore a bound

$$\int_A f = \inf_P \mathcal{U}_P(f) \geq \varepsilon \cdot \eta > 0. \qquad \text{(ECP)}$$

2 For submission to Davis Lazowski

Problem 2.1. If C is a bounded set of measure zero and with integrable characteristic function χ_C, show that the integral $\int_A \chi_C$ is necessarily zero.

Solution. C has measure zero, so for all $\varepsilon > 0$ there exists a sequence of generalised rectangles U_i covering S, such that $\sum_{i=1}^{\infty} \mathrm{vol}(U_i) < \varepsilon$. Therefore $\mathrm{vol}(\bigcup_i U_i) \leq \sum_{i=1}^{\infty} \mathrm{vol}(U_i) < \varepsilon$.

But also $\chi_C \leq \chi_{\bigcup_i U_i}$.

So applying 1.2.1

$$0 \leq \int \chi_C d\mu \leq \int \chi_{\bigcup_i U_i} d\mu \leq \sum_{i=1}^{\infty} \int \chi_{U_i} d\mu = \sum_{i=1}^{\infty} \mathrm{vol}(U_i) < \varepsilon$$

Since this is true for every $\varepsilon > 0$, $\int \chi_C d\mu = 0$. (DL)

Problem 2.2. 1. Show that the collection of all rectangles $[a_1, b_1] \times \cdots \times [a_n, b_n]$ with all a_j and b_j rational (and n fixed) can be arranged into a sequence.

2. Conclude that if $\mathcal{O}$ is an open cover of *any* set $A \subseteq \mathbb{R}^n$, then there is a sequence of opens $U_1, U_2, \ldots \in \mathcal{O}$ chosen from the cover such that $\{U_1, U_2, \ldots\}$ forms a cover of A.[2] (Hint: show that any $U \in \mathcal{O}$ has a rational rectangle in it.)

Solution. 1. We can identify the rectangle $[a_i, b_i]$ with $(a_i, b_i) \subset \mathbb{Q} \times \mathbb{Q}$. In $\mathbb{Q} \times \mathbb{Q}$, we double-count this rectangle, as (a_i, b_i) and (b_i, a_i), but if we we can make a sequence while double-counting we surely can while single counting.

Now by what we did in class there is a bijection $s : \mathbb{Q} \to \mathbb{N}$, which makes $\mathbb{Q}$ a sequence.

Then there is a map

$$(s \times s) : \mathbb{Q} \times \mathbb{Q} \to \mathbb{N} \times \mathbb{N}$$
$$(s \times s)(x, y) \to (s(x), s(y))$$

And a map $c : \mathbb{N} \times \mathbb{N} \to \mathbb{Q} : (x, y) \to \frac{3^x}{2^y}$. This map is clearly injective, by unique factorisation.

Also, we can trivially make a sequence of any infinite subset of $\mathbb{N}$, by letting s_1 be the first element in the subset, s_2 the second, etc. So there is a bijective map $\mathfrak{c} : c(\mathbb{N}) \to \mathbb{N}$.

So there are bijective maps

$$b_\ell : \mathfrak{c} \circ c \circ (s \times s) : \mathbb{Q} \times \mathbb{Q} \to \mathbb{N}$$
$$b_q : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$$

Where the second is induced from b_ℓ by our preexisting bijection $\mathbb{N} \to \mathbb{Q}$.

Applying b_q to each rectangle, we have a bijection

$$(b_q \times b_q \times \ldots b_q) : \mathbb{Q}^{2n} \to \mathbb{Q}^n$$

Now we can apply b_q to the first two copies of $\mathbb{Q}$, and the identity to the rest, to get a bijection $(b_q \times \mathrm{id}) : \mathbb{Q}^n \to \mathbb{Q}^{n-1}$.

Inductively, we have a bijection $\mathbb{Q}^{2n} \to \mathbb{Q} \to \mathbb{N}$, as desired.

2. Every open set has an open ball inside, and an open rectangle inside that. By shrinking, we can make this open rectangle a rational rectangle.

If there was a cover without a countable subcover, there must be an uncountable subcover $\{U_\alpha\}_\alpha$ with the property that for any U_β there is a point, say p_β, so that $p_\beta \notin \{U_\alpha\}_\alpha \setminus \{U_\beta\}$. But around p_β is an open set not included in any other set in the cover, and inside it an open rectangle, which has a rational rectangle inside. So from the p_β we can find uncountably many distinct rational rectangles, which contradicts part 1, so every cover has a countable subcover. (DL)

Problem 2.3. 1. Show that an unbounded set C cannot have content zero.

2. Use this observation to give an example of a closed set of measure zero that is not of content zero.

3. If C is a set of content zero, show that its boundary ∂C has content zero.

4. Exhibit an example of a bounded set C of measure zero such that ∂C does not have content zero.

[2] This condition is sometimes called *second-countability*. Naming schemes in general topology leave a lot to be desired; see also $T_{2\frac{1}{2}}$ *spaces*.

Solution. 1. Suppose it did. Then there is a set of open rectangles, $R_1 ... R_N$, which cover C with volume less than ε. Each rectangle must be bounded; otherwise, it would have infinite volume. Therefore the union of the rectangles is bounded, but it also covers an unbounded set. Contradiction.

2. The set $\mathbb{N}$ has measure zero, because we can let $\mathcal{O}_i = [i - \frac{\varepsilon}{2^{i+1}}, i + \frac{\varepsilon}{2^{i+1}}]$, and $\sum \mathrm{vol}(\mathcal{O}_i) < \varepsilon$. But it is unbounded, so does not have content zero.

3. Let $R_1 ... R_n$ a finite cover of C with total volume less than ε. Then $\partial C \subset \bigcup_i [R_i \cup \partial R_i]$.

   But by definition, R_i is closed so contains ∂R_i. So $\partial C \subset \bigcup_i R_i$. So the same sequence of covers which shows C has content zero shows ∂C has content zero.

4. Take $\mathbb{Q} \cap [0, 1]$. This set has measure zero, but $\partial(\mathbb{Q} \cap [0, 1]) = [0, 1] - \mathbb{Q}$, which does not have measure zero, so does not have content zero. (DL)

3 For submission to Handong Park

Problem 3.1. Let $f\colon A \to \mathbb{R}$ be integrable, and suppose that $g\colon A \to \mathbb{R}$ agrees with f except at finitely many points. Show that g is also integrable (with the same integral).

Solution. Let n be the number of points at which f and g differ, take an enumeration $\{x_1, \ldots, x_n\}$ of the points, and let $|f(x_j) - g(x_j)| < M$ be an upper bound on the magnitude of these differences. For any $\varepsilon > 0$, let P be a partition such that $\mathcal{U}_P f - \mathcal{L}_P f < \varepsilon/3$. Let $\eta = \varepsilon/(6nM)$ be a secondary error bound, and let P' be the partition with the points $x_j - \eta$ and $x_j + \eta$ added. Since P' refines P, it also satisfies $\mathcal{U}_{P'} f - \mathcal{L}_{P'} f < \varepsilon/3$. We now calculate this difference for g:

$$\begin{aligned}\mathcal{U}_{P'} g - \mathcal{L}_{P'} g &= \sum_{S \in P'} (M_g(S) - m_g(S)) \cdot \mathrm{vol}(S) \\ &= \sum_{\substack{S \in P' \\ x_j \notin S}} (M_g(S) - m_g(S)) \cdot \mathrm{vol}(S) + \sum_{j=1}^{n} (M_g(S_j) - m_g(S_j)) \cdot \mathrm{vol}(S_j),\end{aligned}$$

where $S_j \in P'$ is the unique rectangle with $x_j \in S_j$. This then gives

$$\begin{aligned}&\geq \sum_{\substack{S \in P' \\ x_j \notin S}} (M_f(S) - m_f(S)) \cdot \mathrm{vol}(S) + \sum_{j=1}^{n} (M_f(S_j) - m_f(S_j) + 2M) \cdot \mathrm{vol}(S_j) \\ &\geq \sum_{S \in P'} (M_f(S) - m_f(S)) \cdot \mathrm{vol}(S) + (2Mn) \cdot 2\eta > \varepsilon/3 + \varepsilon/3.\end{aligned}$$

It follows that g is integrable with the same integral. (ECP)

Problem 3.2. Let $f, g\colon A \to \mathbb{R}$ both be integrable functions.

1. For any partition P of A and for any subrectangle S of P, show

$$m_S(f) + m_S(g) \leq m_S(f + g), \qquad M_S(f + g) \leq M_S(f) + M_S(g).$$

2. Conclude that $f + g$ is integrable and that $\int_A (f + g) = \int_A f + \int_A g$.

3. For any constant $c \in \mathbb{R}$, show $\int_A cf = c \int_A f$.

Solution. 1. We'll just prove one of them; the other is identical, with the inequalities reversed. Every value of the form $f(x)+g(x)$ for $x \in S$ is a special case of a value of the form $f(x)+g(y)$ for $x, y \in S$. Hence, there is an inclusion

$$\{f(x)+g(x) \mid x \in S\} \subseteq \{f(x)+g(y) \mid x, y \in S\}$$

which is translated to an inequality by taking suprema:

$$M_S(f+g) = \sup\{f(x)+g(x) \mid x \in S\} \leq \sup\{f(x)+g(y) \mid x, y \in S\} = M_S(f) + M_S(g).$$

2. For any $\varepsilon > 0$, let P_f be a partition such that $\mathcal{U}_f(P_f) - \mathcal{L}_f(P_f) < \varepsilon/2$ and let P_g be a partition such that $\mathcal{U}_g(P_g) - \mathcal{L}_g(P_g) < \varepsilon/2$. Take $P = P_f \cup P_g$ to be their common refinement; P then satisfies both of these inequalities simultaneously. We thus have the following chain:

$$\begin{aligned}\mathcal{U}_{f+g}(P) - \mathcal{L}_{f+g}(P) &\leq (\mathcal{U}_f(P) + \mathcal{U}_g(P)) - (\mathcal{L}_f(P) + \mathcal{L}_g(P)) \\ &= (\mathcal{U}_f(P) - \mathcal{L}_f(P)) + (\mathcal{U}_g(P) - \mathcal{L}_g(P)) \\ &< \varepsilon/2 + \varepsilon/2.\end{aligned}$$

It follows that $f+g$ is integrable with the same integral.

3. This is way easier: take $c > 0$, then

$$\mathcal{U}_{cf}(P) = \sum_{S \in P} M_{cf}(S) \cdot \mathrm{vol}(S) = \sum_{S \in P} cM_f(S) \cdot \mathrm{vol}(S) = c\mathcal{U}_f(P).$$

Making the same calculation for the lower sums, any P satisfying $\mathcal{U}_f(P) - \mathcal{L}_f(P) < \varepsilon/c$ will satisfy $\mathcal{U}_{cf}(P) - \mathcal{L}_{cf}(P) < \varepsilon$. (There are wrinkles here when $c < 0$, which reverses the upper and lower sums, or when $c = 0$, where everything is identically zero and any partition will work.) (ECP)

Problem 3.3. Show that if $f, g\colon A \to \mathbb{R}$ are integrable, then so is their product $f \cdot g$.

Solution. The fastest thing to do is to use the identity

$$fg = \frac{1}{2}((f+g)^2 - f^2 - g^2).$$

We know by the previous problems that sums and scalar multiples of integrable functions are integrable. We also know by the first proof of Problem 1.2.2 that the postcomposite of an integrable function with a continuous function (like $x \mapsto x^2$) is integrable. We conclude that each of the pieces of the right-hand side are integrable.

(You can also pick bounds $|f| < \mu_f$ and $|g| < \mu_g$, pick a mutual refinement of a partition of f and a partition of g that break the difference of upper and lower integrals to within $\varepsilon/(2\mu_g)$ and $\varepsilon/(2\mu_f)$ respectively, and break into a large case analysis concerning the signs of f and g on different subrectangles. This is possible, but *much* less pleasant.) (ECP)

4 For submission to Rohil Prasad

Problem 4.1. Show that an increasing function $f : [a, b] \to \mathbb{R}$ is integrable. (Hint: consider how just how little such a function can fail to be continuous.)

Solution. Set a partition $P_n = \{a, a + \frac{b-a}{n}, \ldots, a + (n-1)\frac{b-a}{n}, b\}$ for every n.

Then, let $S_{n,i}$ be the subrectangle of P_n given by the interval $[a + (i-1)\frac{b-a}{n}, a + i\frac{b-a}{n}]$.

Since f is increasing, we calculate

$$m_{S_{n,i}} = \inf f|_{[a+(i-1)\frac{b-a}{n}, a+i\frac{b-a}{n}]} = f\left(a + (i-1)\frac{b-a}{n}\right)$$

and

$$M_{S_{n,i}} = \sup f|_{[a+(i-1)\frac{b-a}{n}, a+i\frac{b-a}{n}]} = f\left(a + i\frac{b-a}{n}\right).$$

From this, we can calculate the lower and upper sums:

$$L_{P_n} = \sum_{i=1}^{n} m_{S_{n,i}} \cdot \text{length}(S_{n,i}) = \sum_{i=1}^{n} f\left(a + (i-1)\frac{b-a}{n}\right) \cdot \frac{b-a}{n},$$
$$U_{P_n} = \sum_{i=1}^{n} M_{S_{n,i}} \cdot \text{length}(S_{n,i}) = \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) \cdot \frac{b-a}{n}.$$

Therefore, we find $U_{P_n} - L_{P_n} = (f(b) - f(a)) \cdot \frac{b-a}{n}$. It follows that $\lim_{n\to\infty} U_{P_n} \to L_{P_n} = 0$ and as a result f must be integrable. (RP)

Problem 4.2. If A is a Jordan-measurable set and $\varepsilon > 0$, show that there is a compact Jordan-measurable set $C \subseteq A$ such that $\int_{A\setminus C} 1 < \varepsilon$

Solution. To accomplish this, we will construct a sequence of compact Jordan-measurable subsets $C_1, C_2, \dots \subset A$ such that $\int_{C_1} 1 < \int_{C_2} 1 < \dots$.

Since the sequence $\int_{C_i} 1$ is then monotonically increasing and bounded by $\int_A$, it follows that for sufficiently large i we will have $\int_{A\setminus C_i} 1 = \int_A 1 - \int_{C_i} 1 < \varepsilon$.

To construct C_i, take a cover of ∂A by taking an open box with side lengths $1/i$ centered at x for every $x \in \partial A$. Since ∂A is compact, this cover has a finite subcover. We then construct C_i by removing the union of all of these open rectangles from A.

Thus, we have a finite cover of ∂A by open boxes of side length $1/i$ for every i. Call these covers $\mathcal{U}_i$. Then, we refine the cover by taking $\mathcal{V}_i$ the cover consisting of pairwise intersections of the sets in $\mathcal{U}_i$ with those in $\cup_{1\le j<i}\mathcal{U}_j$. Since this is a set of pairwise intersections of two finite collections of sets, this is a finite cover as well.

Now we construct $C_i = A \setminus \cup_{R\in\mathcal{V}_i} R$. Note that this is equivalent to $\overline{A} \cap (\bigcap_{R\in\mathcal{V}_i} \mathbb{R}^n \setminus R)$, which is an intersection of finitely many closed sets and as a result is closed itself. Furthermore, C_i is bounded so it is compact.

Furthermore, the intersection of two rectangles is a rectangle, so the boundary of C_i is the union of the boundary of finitely many rectangles. This clearly has measure 0, so C_i is Jordan-measurable.

Finally, we show that $\int_{C_i} 1 < \int_{C_j} 1$ for $i < j$. To see this, observe that $C_j - C_i = \cup_{R\in\mathcal{V}_i} R - \cup_{R\in\mathcal{V}_j} R$. These sets are Jordan-measurable. Therefore, the inequality is equivalent to $\int_{\cup_{R\in\mathcal{V}_j} R} 1 < \int_{\cup_{R\in\mathcal{V}_i} R} 1$. This follows because every set R in $\mathcal{V}_j$ is strictly contained inside some R' in $\mathcal{V}_i$, so there must exist some $R' \in \mathcal{V}_i$ such that $R' - \cup_{R\in\mathcal{V}_j} R$ is nonempty. This set itself is then equal to an open rectangle with finitely many open rectangles removed from its interior, so it has measure greater than 0 as desired.

Now we have that the sequence of $\int_{C_i} 1$ is a monotonically increasing sequence bounded by $\int_A 1$, so $\int_{A\setminus C_i} 1$ is a monotonically decreasing sequence bounded below by 0. Therefore, $\lim_{i\to\infty} \int_{A\setminus C_i} 1 = 0$, which is equivalent to the problem statement. (RP)

Problem 4.3. Let $A \subseteq \mathbb{R}^n$ be a closed rectangle. Show that subset $C \subseteq A$ is Jordan-measurable if and only if for every $\varepsilon > 0$ there exists a partition P of A satisfying

$$\sum_{S \text{ of type I}} \text{vol}(S) - \sum_{S \text{ of type II}} \text{vol}(S) < \varepsilon,$$

where "type I" are those rectangles intersecting C and "type II" are those rectangles contained in C.

Solution. Note that

$$\sum_{S \text{ of type I}} \text{vol}(S) - \sum_{S \text{ of type II}} \text{vol}(S) = \sum_{S \text{ of type I but not type II}} \text{vol}(S).$$

A subrectangle of type I but not type II is by definition a subrectangle that intersects the boundary ∂C of C. Suppose that C is Jordan-measurable. Then, ∂C has content 0, so for any $\varepsilon > 0$ we can construct a union of finitely many open rectangles $\mathcal{S}$ containing ∂C such that $\sum_{S \text{ in } \mathcal{S}} \text{vol}(S) < \varepsilon$, where we split $\mathcal{S}$ into a grid of subrectangles.

Extending these grid-lines, we can obtain a partition P of A whose subrectangles of type I but not type II are exactly those subrectangles in $\mathcal{S}$ and therefore P has our desired property.

Now we show the other direction. In this case, for any $\varepsilon > 0$ we can pick a partition P such that the union of the subrectangles of type I but not type II contains ∂C and has measure $< \varepsilon$. By definition, ∂C has measure zero and therefore C is Jordan-measurable. (RP)