Homework #2

Math 25b

Due: February 15th, 2017

Guidelines:

- As we calibrate the difficulty of the first few assignments, please write the amount of time it takes you to finish the entire assignment (i.e., the sum of all four parts) at the top of each packet you hand in.
- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one, a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!) [1]

1 For submission to Thayer Anderson

Problem 1.1. A function $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^\ell$ is *bilinear* when it satisfies

$$\begin{aligned} f(cx, y) &= f(x, cy) = cf(x, y), \\ f(x + x', y) &= f(x, y) + f(x', y), \\ f(x, y + y') &= f(x, y) + f(x, y'). \end{aligned}$$

1. Show that if f is bilinear, then
$$\lim_{(h,k)\to 0} \frac{\|f(h, k)\|}{\|(h, k)\|} = 0.$$
2. Prove $D_{(a,b)}f(x, y) = f(a, y) + f(x, b)$.
3. Conclude from this the product rule for functions with target $\mathbb{R}$.

Problem 1.2. Here's the same problem with more indices. Consider the product $V = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$ of Euclidean spaces of various dimensions. A function $f\colon V \to \mathbb{R}^\ell$ is called *multilinear* when for any choice of tuple of vectors $(v_1, \ldots, v_m)$, the function

$$\begin{aligned} f_j\colon \mathbb{R}^{n_j} &\to \mathbb{R}^\ell, \\ f_j(v) &= f(v_1, \ldots, v_{j-1}, v, v_{j+1}, \ldots, v_m) \end{aligned}$$

describes a linear function.

[1] This version of the homework dates from February 13, 2017.

1. For any point $(a_1, \ldots, a_m) \in V$, difference vector $(h_1, \ldots, h_m) \in V \cong T_a V$, and pair of coordinates $i < j$, show the following:

$$\lim_{\|h\| \to 0} \frac{\|f(a_1, \ldots, a_{i-1}, h_i, a_{i+1}, \ldots, a_{j-1}, h_j, a_{j+1}, \ldots, a_m)\|}{\|h\|} = 0.$$

2. Conclude that the derivative of f is given by

$$(D_a f)(h) = \sum_{j=1}^{m} f(a_1, \ldots, a_{j-1}, h_j, a_{j+1}, \ldots, a_m).$$

2 For submission to Davis Lazowski

Problem 2.1. In this problem, we show that C^∞ functions are generally *not* determined by their Taylor expansion.[2]

1. Define $f_1 \colon \mathbb{R} \to \mathbb{R}$ by the following piecewise formula:

$$f_1(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

   Show that f_1 is a C^∞ function and that $f_1^{(j)}(0) = 0$ for all orders j.

2. Now define $f_2 \colon \mathbb{R} \to \mathbb{R}$ by

$$f_2(x) = \begin{cases} e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}} & \text{if } x \in (-1, 1), \\ 0 & \text{otherwise.} \end{cases}$$

   Show that f_2 is a C^∞ function which is positive on $(-1, 1)$ and zero elsewhere.

3. For any $\varepsilon > 0$, show that there is a C^∞ function $g \colon \mathbb{R} \to \mathbb{R}$ such that $g(x \leq 0) = 0$, $g(x \geq \varepsilon) = 1$, and $0 \leq g(x) \leq 1$. (If you want, you're welcome to use the single-variable fundamental theorem of calculus here. You're not obligated, of course.)

Problem 2.2. We now use the functions from the previous problem to construct interesting functions on $\mathbb{R}^n$.

1. Let f_2 be as above, and select any $\varepsilon > 0$ and point $(a_1, \ldots, a_n) \in \mathbb{R}^n$. Define a function $h \colon \mathbb{R}^n \to \mathbb{R}$ by the formula

$$h(x_1, \ldots, x_n) = \prod_j f_2\left(\frac{x_j - a_j}{\varepsilon}\right).$$

   Show that h is a C^∞ function which is positive on the open rectangle $\prod_j (a_j - \varepsilon, a_j + \varepsilon)$ and zero elsewhere.

2. For $A \subseteq \mathbb{R}^n$ open and $C \subseteq A$ compact, show there is a nonnegative C^∞ function $f \colon A \to \mathbb{R}$ such that $f(x \in C) > 0$, and f becomes the zero function outside of some closed set contained in A.

3. Show we can choose the f from the previous part so that $0 \leq f(x) \leq 1$ and and $f(x \in C) = 1$.

[2] This stronger condition is sometimes called *analyticity*, and the function is called *analytic*.

3 For submission to Handong Park

Let's keep it going from Thayer's part with some applications.

Problem 3.1. Recall that inner products give examples of bilinear functions. Define $IP\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by the following formula:

$$IP(x, y) = \langle x, y \rangle,$$

where $\langle -, - \rangle$ is the usual Euclidean inner product.

1. Describe $D_{(a,b)} IP$.
2. Given two differentiable functions $f, g\colon \mathbb{R} \to \mathbb{R}^n$, form the composite $h(t) = \langle f(t), g(t) \rangle$. For any $t_0 \in \mathbb{R}$, show
$$h'(t_0) = \langle (D_{t_0} f)(t), g(t_0) \rangle + \langle f(t_0), (D_{t_0} g)(t) \rangle,$$
where we are identifying the $(n \times 1)$–matrix $D_{t_0} f$ with a column vector, or an element in $\mathbb{R}^n$.
3. For $f\colon \mathbb{R} \to \mathbb{R}^n$ differentiable and $\|f(t)\| = 1$ for all $t \in \mathbb{R}$, conclude that for any $t_0 \in \mathbb{R}$ we have
$$\langle (D_{t_0} f)(t), f(t_0) \rangle = 0.$$

4 For submission to Rohil Prasad

Problem 4.1. We now use this technology to calculate the derivative of the determinant. Use the columns of an $(n \times n)$ matrix M to present it as an element as

$$M \in \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{n \text{ copies}}.$$

1. Demonstrate $(D_a \det)(v) = \sum_{j=1}^n \det\left(a_1 | \cdots | a_{j-1} | v_j | a_{j+1} | \cdots | a_n\right)$.
2. Assemble a collection of differentiable functions $a_{ij}\colon \mathbb{R} \to \mathbb{R}$ into a matrix $(a_{ij})_{ij}$, and set $f(t) = \det(a_{ij}(t))$. Calculate its derivative to be
$$(D_{t_0} f)(h) = \sum_{j=1}^n \det \left(\begin{array}{c|c|c|c|c|c|c} a_{11}(t_0) & \cdots & a_{1(j-1)}(t_0) & a'_{1j}(t_0) \cdot h & a_{1(j+1)}(t_0) & \cdots & a_{1n}(t_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t_0) & \cdots & a_{n(j-1)}(t_0) & a'_{nj}(t_0) \cdot h & a_{n(j+1)}(t_0) & \cdots & a_{nn}(t_0) \end{array} \right).$$
3. Take f as in the previous part, and suppose $f(t) \neq 0$ for all t. Given $b_1, \ldots, b_n\colon \mathbb{R} \to \mathbb{R}$ differentiable, let $x_1, \ldots, x_n\colon \mathbb{R} \to \mathbb{R}$ be the functions satisfying the matrix equation
$$(a_{ij}(t))_{ij} \cdot (x_i(t))_i = (b_i(t))_i$$
guaranteed by the invertibility of $(a_{ij}(t))_{ij}$. Show that the functions x_i are all differentiable and calculate their derivatives.