Homework #1

Math 25b

Due: February 8th, 2017

Guidelines:

- As we calibrate the difficulty of the first few assignments, please write the amount of time it takes you to finish the entire assignment (i.e., the sum of all four parts) at the top of each packet you hand in.
- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one, a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!)[1]

1 For submission to Thayer Anderson

Problem 1.1. (Linear algebra review:) For $x \in \mathbb{R}^n$, write $x = x_1 e_1 + \cdots + x_n e_n$ in some orthonormal basis. Show $\|x\| \leq \sum_{j=1}^n \|x_j\|$.

Problem 1.2.

1. (Linear algebra review:) For a linear function $f\colon \mathbb{R}^n \to \mathbb{R}^m$, show there exists a real number M such that $\|f(v)\| \leq M\|v\|$ for all v. (Note that M is not allowed to depend upon v.)
2. Show that a linear function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is automatically continuous.

Problem 1.3. Show that if $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then it is also continuous at a.

2 For submission to Davis Lazowski

Problem 2.1. For any subset $A \subseteq \mathbb{R}^n$ which is not closed, show that there exists a continuous function $f\colon A \to \mathbb{R}$ which is not bounded.

Problem 2.2. Suppose that $A \subseteq \mathbb{R}^n$ is a closed set, that $B \subseteq \mathbb{R}^n$ is a compact set, and that $A \cap B = \emptyset$.

1. For fixed $y \in \mathbb{R}^n \setminus A$, show there exists a real value $d > 0$ such that for any $x \in A$, $\|x - y\| \geq d$.
2. Show that every continuous function $f\colon B \to \mathbb{R}$ achieves a minimum and maximum value.

[1] This version of the homework was posted on February 3, 2017.

3. Show that there exists a real value $d > 0$ such that for *any* $x \in A$ and $y \in B$, we have $\|x - y\| \geq d$.
4. Show that parts 2 and 3 both fail in $\mathbb{R}^2$ if we merely ask B to be closed but not compact.

Problem 2.3. Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}$ satisfies $|f(x)| \leq \|x\|^2$. Show that f is automatically differentiable at 0.

3 For submission to Handong Park

Problem 3.1.
1. Let $\mathcal{F}$ be a family of open sets, possibly infinite in length. Show that the union $\bigcup_{U \in \mathcal{F}} U$ is again an open set.
2. Let U and V be two individual open sets. Show that $U \cap V$ is again an open set.
3. Show that there exists a family $\mathcal{F}$ of open sets such that $\bigcap_{U \in \mathcal{F}} U$ is *not* an open set.

Problem 3.2. Let $U \neq \emptyset$ be an open subset of $\mathbb{R}^n$ and let $C \subseteq U$ be compact. Show that there exists a compact set D such that

$$C \subseteq D^\circ \subseteq D \subseteq U,$$

where D° denote the interior of D.

Problem 3.3. Let $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ be a function $f\colon \mathbb{R} \to \mathbb{R}^2$. Show that f is differentiable at a if and only if f_1 and f_2 both are, and in that case then $D_a f = \begin{pmatrix} D_a f_1 \\ D_a f_2 \end{pmatrix}$.

4 For submission to Rohil Prasad

Problem 4.1. Show that if A is closed and $[0,1] \cap \mathbb{Q} \subseteq A$, then actually $[0,1] \subseteq A$.

Problem 4.2. Let A denote the subset $A = \{(x,y) \in \mathbb{R}^2 \mid x > 0, 0 < y < x^2\}$, and let $\chi_A\colon \mathbb{R}^2 \to \mathbb{R}$ denote the *indicator function*

$$\chi_A(x,y) = \begin{cases} 1 & \text{if } (x,y) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

1. Let L be any line through the origin. Show that there is a neighborhood of the origin in L that has no intersection with A.
2. For each $h \in \mathbb{R}^2$, let $g_h\colon \mathbb{R} \to \mathbb{R}^2$ parametrize a line by

   $$g_h(t) = h \cdot t.$$

   Show that $\chi_A \circ g_h$ always defines a continuous function at the origin.
3. Nonetheless, show that χ_A is not a continuous function. (Exhibit a continuous curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ with $\gamma(0) = (0,0)$ such that $\chi_A \circ \gamma$ is not a continuous function.)

Problem 4.3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be an arbitrarily differentiable function on the real line, and let $P^n_{f@a}(x)$ denote the n^{th} order Taylor polynomial of f:

$$P^n_{f@a}(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a) \cdot (x-a)^j}{j!}.$$

Show that f and $T^n_{f@a}$ agree to n^{th} order a, i.e.,

$$\lim_{h \to 0} \frac{f(a+h) - T^n_{f@a}(a+h)}{h^n} = 0.$$

(Feel free to use tools you know from calculus to evaluate this limit.)