Homework #1 Solutions

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1 For submission to Thayer Anderson

Problem 1.1. For $x \in \mathbb{R}^n$, write $x = x_1 e_1 + \cdots + x_n e_n$ in some orthonormal basis. Show $||x|| \leq \sum_{j=1}^n ||x_j||$.

Solution. By definition, $||x||^2 = \langle x, x \rangle$. Then manipulate as follows:

$$\begin{aligned}
|x||^2 = \langle x, x \rangle &= \left\langle \sum_{j=1}^n ||x_j||, \sum_{j=1}^n ||x_j|| \right\rangle \\
&= \sum_{j=1}^n x_j \left\langle e_i, \sum_{j=1}^n ||x_j|| \right\rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n x_j \overline{x_j} \langle e_i, e_j \rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n x_j \overline{x_j} \delta_i(j) = \sum_{j=1}^n ||x_j||^2.
\end{aligned}$$

And then because $a^2 + b^2 \leq (a+b)^2$ for $a, b > 0$, the desired result is obtained. (TA)

Problem 1.2. 1. For a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$, show there exists a real number M such that $||f(v)|| \leq M||v||$ for all v. (Note that M is not allowed to depend on v).

2. Show that a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ is automatically continuous.

Solution. 1. Fix an orthonormal basis for $\mathbb{R}^n$, $e_1, \ldots, e_n$. Let $S = \{||f(v)|| : ||v|| = 1\}$. I claim that S is bounded. To show this, I exhibit the following bound. Let $B = \max\{||f(e_1)||, \ldots, ||f(e_n)||\}$. As the maximum of a finite set of numbers, B is exists. Then suppose $v = x_1 e_1 + \cdots + x_n e_n$ arbitrary such that $||v|| = 1$. Then

$$||f(v)|| = ||x_1 f(e_1) + \cdots + x_n f(e_n)|| \leq \sum ||x_i|| \leq n \cdot B$$

by Problem 1. Thus $n \cdot B$ bounds S from above. Then $M := \sup(S)$ exists. Take v arbitrary. Then $||f(v)|| \leq M||v||$. To see this, suppose the contrary, namely that there exists a vector v with $||f(v)|| > M||v||$. then we see

$$||f(v/||v||)|| = \frac{1}{||v||}||f(v)|| > M \cdot \frac{||v||}{||v||} = M$$

In that case, we see that the vector $v/||v||$ has norm 1 and $||f(v/||v||)||$ exceeds the upper bound M. This is a contradiction.

2. To show that f is continuous, we show that it is continuous at every point. Fix a vector v. Suppose that $\varepsilon > 0$ arbitrary. Let M be as above with respect to the function f. Then take $\delta = \varepsilon/2M$. Suppose that u satisfies $d(u, v) < \delta$. Then

$$d(f(u), f(v)) = ||f(u) - f(v)|| = ||f(u - v)|| \leq M \cdot ||u - v|| < \varepsilon/2$$

and we are done. (TA)

Problem 1.3. Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then it is also continuous at a.

Solution. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function differentiable at a. Suppose for the sake of contradiction that f is not continuous at a. Then there exists some fixed $\varepsilon_0 > 0$ such that there does not exist any $\delta > 0$ satisfying $||x - a|| < \delta \Rightarrow ||f(x) - f(a)|| < \varepsilon$. By differentiability of f at a, there exists a δ_0 such that $||x - a|| < \delta_0$ implies

$$\frac{||f(x) - f(a)||}{||x - a||} < \varepsilon_0$$

(note this is the same ε_0 as fixed above.) Then taking $\delta = \min\{1, \delta_0\}$. We see that $||x - a|| < \delta_0$ implies that $||f(x) - f(a)|| < \varepsilon$ which is a contradiction. (TA)

2 For submission to Davis Lazowski

Problem 2.1. For any subset $A \subseteq \mathbb{R}^n$ which is not closed, show that there exists a continuous function $f\colon A \to \mathbb{R}$ which is not bounded.

Solution. Then there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ in A which converges in $\mathbb{R}^n$ but does not converge in A. Let $\ell := \lim_{n\to\infty} a_n$.

Define $f_\ell : A \to \mathbb{R}$ as $f_\ell(x) := \frac{1}{||\ell - x||}$.

Claim. f_ℓ is not bounded on A.

If f_ℓ were bounded on A, it would be bounded on the sequence a_n. So there would be some C such that

$$\frac{1}{||\ell - a_n||} = f_\ell(a_n) \leq C < \infty$$
$$\iff 0 < \frac{1}{C} \leq ||\ell - a_n||$$

Therefore the a_n cannot converge to ℓ, therefore contradiction.

Claim. f_ℓ is continuous.

Importantly, this is with respect to the metric on A: f_ℓ isn't even well defined on $\mathbb{R}^n$.

Let $\{b_n\}$ be a sequence in A which converges to b. Then $||b_n|| \to ||b||$, and by translation invariance $||b_n - \ell|| \to ||b - \ell||$. Therefore $\frac{1}{||b_n - \ell||} \to \frac{1}{||b - \ell||}$, as desired. (DL)

Problem 2.2. Suppose that $A \subseteq \mathbb{R}^n$ is a closed set, that $B \subseteq \mathbb{R}^n$ is a compact set, and that $A \cap B = \emptyset$.

1. For fixed $y \in \mathbb{R}^n \setminus A$, show there exists a real value $d > 0$ such that for any $x \in A$, $\|x - y\| \geq d$.
2. Show that every continuous function $f\colon B \to \mathbb{R}$ achieves a minimum and maximum value.
3. Show that there exists a real value $d > 0$ such that for *any* $x \in A$ and $y \in B$, we have $\|x - y\| \geq d$.
4. Show that parts 2 and 3 both fail in $\mathbb{R}^2$ if we merely ask B to be closed but not compact.

Solution.

1. Otherwise, we can find a sequence x_n so that $||x_n - y|| < \frac{1}{n}$. The sequence x_n converges to y. But A is closed and $y \notin A$, therefore contradiction.

2. $f(B)$ is compact, so by Heine-Borel is closed and bounded. Its upper bound will be the maximum; its lower bound will be the minimum.

3. Since B is bounded, there is some closed interval containing it – say $[-b, b]$. Enlarge this closed interval to $[-b-\varepsilon, b+\varepsilon]$.

   Now $\tilde{A} := A \cap [-b-\varepsilon, b+\varepsilon]$ is a closed and bounded set, so is compact. $\tilde{A}$ also includes every point in A which is in at least ε of B. So if $\tilde{A} = \emptyset$, take $\varepsilon := d$.

   Now we can define a function $f : \tilde{A} \times B \to \mathbb{R}$ by $f(a, b) = ||a - b||$. This function is continuous, and $\tilde{A} \times B$ is compact. Therefore $f(\tilde{A} \times B)$ is compact, so achieves a minimum and maximum. Call this minimum d. Now $d > 0$, because for every a the image $f(a, B)$ is compact, so has a minimum ℓ_a, with $\ell_a > 0$ by part a), so $0 \notin f(\tilde{A} \times B)$.

4. If B is closed, there are counterexamples to 2). For example, take $f = \text{id}$.

   There are also counterexamples to 3). Take

$$A := \bigcup_{j \in \mathbb{Z}} [j, j + \frac{1}{2}]$$
$$B := \bigcup_{j \in \mathbb{Z}, |j| > 4} [j + \frac{1}{2} + \frac{1}{|j|}, j + 0.9]$$

   Then we can find $j + \frac{1}{2} \in A$, $j + \frac{1}{2} + \frac{1}{|j|} \in B$, so that the distance between these points is $\frac{1}{|j|}$, which converges to 0 as $j \to \infty$. (DL)

Problem 2.3. Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}$ satisfies $|f(x)| \leq \|x\|^2$. Show that f is automatically differentiable at 0.

Solution. We need to find some $D_0 f$ such that

$$f(h) = f(0) + D_0 f(h) + \varepsilon(h)$$

And

$$\lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$$

Suppose such a $D_0 f$ existed. We have

$$\begin{aligned} |\varepsilon(h)| &= |f(h) - f(0) - D_0 f(h)| \\ &\leq ||h||^2 + |D_0 f(h)| \end{aligned}$$

So

$$\frac{|\varepsilon(h)|}{||h||} \leq ||h|| + \frac{|D_0 f(h)|}{||h||}$$

If $D_0 f$ were not zero, then $\frac{|D_0 f(h)|}{||h||}$ would not converge to zero. This suggests that we guess $D_0 f = 0$. An alternative motivation for this guess could be: the derivative is supposed to be the linear change at a given point; but our bound tells us that there is only second order change in f at 0.

Then we can verify

$$\lim_{h \to 0} \frac{|f(h) - f(0)|}{||h||} = \lim_{h \to 0} \frac{|f(h)|}{||h||} \leq \lim_{h \to 0} \frac{||h||^2}{||h||} \leq \lim_{h \to 0} ||h|| = 0 \qquad \text{(DL)}$$

3 For submission to Handong Park

Problem 3.1. 1. Let $\mathcal{F}$ be a family of open sets, possibly infinite in length. Show that the union $\bigcup_{U\in\mathcal{F}} U$ is again an open set.

2. Let U and V be two individual open sets. Show that $U \cap V$ is again an open set.

3. Show that there exists a family $\mathcal{F}$ of open sets such that $\bigcap_{U\in\mathcal{F}} U$ is *not* an open set.

Solution. 1. Call the union $M = \bigcup_{U\in\mathcal{F}} U$. Given any point m in M, we must show that there exists $R > 0$ such that $B_R(m) \in M$. But for any $m \in M$, $m \in U$ for one of the open sets $U \in F$. Since each U is open, exists R_U such that $B_{R_U}(m) \subset U$ is completely contained. Since $U \subset M$, $B_{R_U}(m) \subset M$ is conpletely contained, and we are done.

2. Given U and V open, and any $x \in U \cap V$, there exists $R_U > 0$ such that $B_{R_U}(x) \subset U$ and $R_V > 0$ such that $B_{R_V}(x) \subset V$. Take the minimum of R_U and R_V, call this R_{min}. Then $B_{R_{min}}(x) \subset U$ and $B_{R_{min}}(x) \subset V$, so $B_{R_{min}}(x) \subset U \cap V$, showing openness.

3. There are several examples that work. One simple one is to take the set of all $(-\frac{1}{n}, \frac{1}{n})$ open intervals for $n \in \mathbb{N}$ and intersect them, giving us the closed set $[0, 0]$, which is not open. (HP)

Problem 3.2. Let $U \neq \emptyset$ be an open subset of $\mathbb{R}^n$ and let $C \subseteq U$ be compact. Show that there exists a compact set D such that

$$C \subseteq D^\circ \subseteq D \subseteq U,$$

where D° denote the interior of D.

Solution. First off, we know that since U is open, $U^C \subset \mathbb{R}^n$ must be closed. We know that by part 2.2.3 of this homework, there must exist some $d > 0 \in \mathbb{R}$ such that given any $x \in U^C$ and $y \in C$, $||x - y|| \geq d$.
We can then use the fact that C is compact by considering the following open cover of C - for each $c \in C$, take $B_{0.5}(c)$. Then, the union of all these open balls gives us an open cover of C, since each $c \in C$ is contained. But since C is compact, only a finite subcover is needed - call this finite subcover $D^\circ = U_1, ..., U_n$, where each U_i is one of the open balls.
Then consider the cover given by $\bar{U_1}, ..., \bar{U_n}$ where each $\bar{U_i}$ is the closure of the corresponding U_i - this is still a cover of C since all we've done is include more points (namely the boundary of each ball). Call this closed ball cover D.
D is completely contained in U since these balls have radius $\frac{d}{2}$, while the minimum distance between any point in C and U^C is d as stated above. Since D is just the closure of D°, we get

$$C \subset D^\circ \subset D \subset U$$

as desired. (HP)

Problem 3.3. Let $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ be a function $f\colon \mathbb{R} \to \mathbb{R}^2$. Show that f is differentiable at a if and only if f_1 and f_2 both are, and in that case then $D_a f = \begin{pmatrix} D_a f_1 \\ D_a f_2 \end{pmatrix}$.

Solution. First, we must prove that if f is differentiable at a, then f_1 and f_2 are also both differentiable and $D_a f$ is the desired expression. $D_a f$ is just a linear operator, so let's call its linear components $(D_a)_1$ and

$(D_a)_2$. So f's differentiability and the definition of f yields:

$$\lim_{h\to 0} \frac{||f(a+h) - f(a) - D_a f(h)||}{||h||} = 0$$

$$\lim_{h\to 0} \frac{\left|\left| \begin{array}{c} f_1(a+h) \\ f_2(a+h) \end{array} - \begin{array}{c} f_1(a) \\ f_2(a) \end{array} - \begin{array}{c} (D_a)_1(h) \\ (D_a)_2(h) \end{array} \right|\right|}{||h||} = 0$$

$$\lim_{h\to 0} \frac{\left|\left| \begin{array}{c} f_1(a+h) - f_1(a) - (D_a)_1(h) \\ f_2(a+h) - f_2(a) - (D_a)_2(h) \end{array} \right|\right|}{||h||} = 0$$

From here, we just work with the norms and redistribute $||h|| = \sqrt{h^2}$ to get

$$\lim_{h\to} \sqrt{\frac{(f_1(a+h) - f_1(a) - (D_a)_1(h))^2}{h^2} + \frac{(f_2(a+h) - f_2(a) - (D_a)_2(h))^2}{h^2}} = 0$$

Since we have two squared numbers that add to 0, we must have that each individual number is 0, which is only true if

$$\lim_{h\to 0} \frac{f_1(a+h) - f_1(a) - (D_a)_1(h)}{h} = 0$$

meaning that f_1 is differentiable at a, and if

$$\lim_{h\to 0} \frac{f_2(a+h) - f_2(a) - (D_a)_2(h)}{h} = 0$$

meaning that f_2 is also differentiable at a. In addition, these two are just the derivative definitions - by uniqueness of derivatives, we must have that $(D_a)_1$ was indeed $D_a f_1$ and that $(D_a)_2$ was in tact $D_a f_2$.
Now, we will prove that if f_1 and f_2 are both differentiable at a and $D_a f$ is the desired expression, then f is differentiable at a. This time, we must prove that

$$\lim_{h\to 0} \frac{||f(a+h) - f(a) - D_a f(h)||}{||h||} = 0$$

Working through the exact same calculations as above, we again arrive at the expression

$$\lim_{h\to} \sqrt{\frac{(f_1(a+h) - f_1(a) - (D_a)_1(h))^2}{h^2} + \frac{(f_2(a+h) - f_2(a) - (D_a)_2(h))^2}{h^2}} = 0$$

this time with the hope that the equals sign is true. But the equals sign must hold, since the first term is 0 by f_1's differentiability and the second term is 0 by f_2's differentiability. (HP)

4 For submission to Rohil Prasad

Problem 4.1. Show that if A is closed and $[0,1] \cap \mathbb{Q} \subseteq A$, then actually $[0,1] \subseteq A$.

Solution. Assume for the sake of contradiction that there is some irrational $r \in [0,1]$ which does not lie in A. Since A is closed, the complement of A is open.

Therefore, there exists some $\varepsilon > 0$ such that the interval $(r-\varepsilon, r+\varepsilon)$ does not intersect A. However, for any $\varepsilon > 0$ there is a rational number q in $(r-\varepsilon, r+\varepsilon)$. For small enough ε, we have $(r-\varepsilon, r+\varepsilon) \subset [0,1]$ so it follows that any open interval around r intersects $[0,1] \cap \mathbb{Q}$.

As a result, any open interval around r intersects A and we arrive at a contradiction, so r must lie in A. (RP)

Problem 4.2. Let A denote the subset $\{(x,y) \in \mathbb{R}^2 | x > 0, 0 < y < x^2\}$, and let $\chi_A : \mathbb{R}^2 \to \mathbb{R}$ denote the *indicator function*

$$\chi_A(x,y) = \begin{cases} 1 & \text{if } (x,y) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

1. Let L be any line through the origin. Show that there is a neighborhood of the origin in L that has no intersection with A.

2. For each $h \in \mathbb{R}^2$, let $g_h : \mathbb{R} \to \mathbb{R}^2$ parameterize a line by

$$g_h(t) = h \cdot t$$

Show that $\chi_A \circ g_h$ always defines a continuous function at the origin.

3. Nonetheless, show that χ_A is not a continuous function. (Exhibit a continuous curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ with $\gamma(0) = (0,0)$ such that $\chi_A \circ \gamma$ is not a continuous function.)

Solution. 1. Let L be given by (tx, ty) for $t \in \mathbb{R}$ with either x or y being greater than or equal to zero. It is clear that L does not intersect A for $t \le 0$, so the problem statement is equivalent to showing that there exists some $\varepsilon > 0$ such that either $tx \le 0$, $ty \le 0$, or $ty \ge t^2x^2$ for all $t \in (0, \varepsilon)$.

If either $x < 0$ or $y < 0$ then it is clear that tx and ty are respectively negative for all t.

Thus, we have reduced to the case where both x, y are positive. In this case, we have $ty \ge t^2x^2$ is equivalent to $y/x^2 \ge t$. Therefore, we can pick $\varepsilon = y/x^2$ in this case.

2. To show continuity at the origin, we must show for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|t| < \delta$ implies $|(\chi_A \circ g_h)(t)| < \varepsilon$. Since χ_A is either equal to 0 or 1, it suffices to show that there exists $\delta > 0$ such that $|t| < \delta$ implies $(\chi_A \circ g_h)(t) = 0$. The latter statement is equivalent to $g_h(t) \notin A$. Since g_h is a line through the origin, we now have the problem statement is equivalent to showing there is a neighborhood of the origin in g_h that does not intersect A. This is exactly the result of part 1.

3. If χ_A were continuous, then since the composition of continuous functions is continuous the function $\chi_A \circ \gamma$ for γ continuous would be continuous as well.

   Therefore, showing it is not continuous amounts to showing $\chi_A \circ \gamma$ is not continuous.

   We can set γ to send t to $(t,t) \in \mathbb{R}^2$. For $t \in (-\infty, 0]$ it is clear that $\gamma(t) \notin A$. For $t \in (0,1]$ we have $t \ge t^2$, so we have $\gamma(t) \notin A$ in this case as well. For $t \in (1, \infty)$ we have both that $t > 0$ and $t < t^2$, so then $\gamma(t) \in A$.

   Therefore, $\chi_A \circ \gamma$ is the function that is 0 on $(-\infty, 1]$ and 1 on $(1, \infty)$. It is immediate that it is then discontinuous at 1. (RP)

Problem 4.3. Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrarily differentiable function on the real line, and let $P^n_{f@a}(x)$ denote the n^{th} order Taylor polynomial of f:

$$P^n_{f@a}(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a) \cdot (x-a)^j}{j!}.$$

Show that f and $P^n_{f@a}$ agree to n^{th} order a, i.e.,

$$\lim_{h \to 0} \frac{f(a+h) - P^n_{f@a}(a+h)}{h^n} = 0$$

(Feel free to use tools you know from calculus to evaluate this limit.)

Solution. We will prove this by induction. For $n = 0$, we have $P^0_{f@a}(x) = f(a)$ and the statement is immediate.

Now assume that the statement holds for any smooth function f up to $(n-1)$th order.

From plugging in $h = 0$ into the limit, it is clear that both the top and bottom evaluate to 0. Therefore, we can apply L'Hopital's rule to obtain

$$\lim_{h\to 0} \frac{f(a+h) - P^n_{f@a}(a+h)}{h^n} = \lim_{h\to 0} \frac{f'(a+h) - \sum_{j=1}^{n} \frac{f^{(j)}(a)\cdot h^{j-1}}{(j-1)!}}{nh^{n-1}}$$

However, by definition the latter expression is just equal to $\frac{f'(a+h)-P^{n-1}_{f'@a}(a+h)}{nh^{n-1}}$, which by our inductive assumption equals 0 as desired. (RP)