TOPICS FOR TERM PAPERS

MATH 231B

1. Midterm topics

The goal of these midterm papers is to produce nice little expository articles about this or that aspects of algebraic topology. This is good for you — you'll learn something new that you probably wouldn't have encountered otherwise — and it is good for a broader mathematical audience — you'll have done the significant task of digesting and summarizing some difficult topic for them.

The midterm paper should be shorter — probably 5ish pages, or maybe up to 10. Accordingly, these topics may be too broad, so you should feel free as you read about them to just select a few slivers of it to write about.

Your *primary* goal in writing should be to be understandable, interesting, and engaging. If someone wants all the details, that's what your bibliography is for. If you the details fit into your summary too, without detracting from its readability, then all the better!

Important dates:

- Topic due: Wednesday, March 1st
- Rough draft due: Monday, March 13th
- Paper due: Monday, March 20th

Before these deadlines pass, I expect you to email me (at `ecp@math.harvard.edu`) a PDF of your write-up, as well as its TeX source. I'll give you feedback on your rough draft so that you can make edits and improvements, and then your final draft is what I'll actually take for a grade. The topic deadline is just to make sure you aren't blindsided by having to suddenly write a paper after procrastinating until right up to the deadline.

Here are some potential topics:

(1) We skipped a lot of point-set stuff from Chapter 6, and some of it is genuinely interesting. For instance, you could write about the Blakers–Massey theorem (which one can show holds in great generality). There are a lot of examples along these lines.
(2) Sullivan has a famous set of 1970s notes ("the MIT notes"), which cover a *lot* of topics. At this point in the course, the first couple of chapters about the Hasse square for spaces should be accessible to you.
(3) We skipped some of the stuff from the tail of the Brown representability chapter (Ch 9), where Switzer discusses the representability of functors defined just on finite complexes. There are lots of interesting wrinkles to this theory, including the presence of "phantom maps", and there's a fun article by Gray and McGibbon giving a universal example of such a thing.
(4) Topological data analysis (e.g., persistent homology, motion planning, ...) is a big selling point of algberaic topology to applied mathematicians these days. You could learn about some of these methods.
(5) Morse theory gives an interesting geometric presentation of real ordinary cohomology. Remarkably, there are also variations of this that work for ordinary cohomology with coefficients, and everything about ordinary cohomology can be lensed through to the land of Morse functions. The story for Steenrod operations seems especially interesting. (This might be a final project instead.)
(6) Vector bundles and spherical bundles give an example of an extraordinary cohomology theory, called K– and J–theory respectively. Just setting this up and learning about the theory of structured fiber bundles could be fun, but then there are lots of theorems you could try to prove once you've accomplished the set-up (e.g., Bott periodicity).
(7) Co/bordism also forms an example of an extraordinary cohomology theory, and it is exceptionally interesting. I recommend learning about Kochman's chain functor, but then there are lots of theorems you could

learn about from here: Pontryagin–Thom, for something basic, or any of the more difficult applications to manifold geometry.

(8) Covering spaces find application in algebra through Galois theory. You coudl learn about Andreas Dress's modern proof of the Galois correspondence, to start. Or, you could learn about étale maps, the algebraic analogue of covering spaces.

(9) We've spent a lot of time talking about representable functors. Hopf algebras form the cogroup objects in k–algebras, and they have an extremely rich theory. For instance, you could learn about the Borel–Milnor–Moore theorem (from the ultra-famous article by Milnor and Moore).

(10) Quasicategories are a big deal at this university, and they are a very technically-advantageous instantiation of the theory of categories enriched over spaces (which is where all our homotopy theory from class is happening). You can automatically learn a lot about the lay of the land of algebraic topology by trying to learn some about these ∞–categorical things. Trying to understand what homotopy co/limits are is, in particular, a *very* worthy goal.

(11) Similarly, you could learn about the Dwyer–Kan hammock localization of a category at a class of weak equivalences, and perhaps the relationship between this construction and the theory of model categories.

(12) There's an old and odd theorem of McDuff, published in a paper titled *On the classifying spaces of discrete monoids*, which shows that every connected space has the homotopy type of the nerve of a one-object category with a finite endomorphism monoid. Wrapping your head around any part of this theorem would be fun. You'll certainly run into the topological notion of "group completion" by thinking about this, which is also worth understanding.

(13) In a different direction, every finite CW complex has the weak homotopy type of a space with finitely many points. As a basic example, look up what a "quasicircle" is.

(14) One of the consequences of the simplicial approximation theorem, together with the Serre finiteness theorem (which we have not yet proven, but which we hope to do eventually), is that the entire homotopy theory of spaces can be done with finite combinatorics. One model for this is *simplicial sets*, and they are so useful that one often inserts homotopy theory into other fields than algebraic topology by doing all the usual constructions but with simplicial sets in place of sets. (For instance, there's the Dold–Kan correspondence in homological algebra, or there's Quillen's use of simplicial rings in setting up his cohomology of rings.) It would be a good use of time to learn about these things.

(15) More basically, there's all of homological algebra! If you don't know much homological algebra, there's a *lot* to learn here that ranges from the basic to the advanced: derived functors, Tor–dimension of rings, algebraic Koszul duality, ...

(16) Classical work of Peter May describes the operations that act on a(n iterated) loopspace in terms of a structure he calls an "operad". You could try to understand what an E_n–space is and how one can recover a "delooping" from that data.

(17) A very famous theorem from topology, the Hopf invariant 1 theorem, states that there are only a handful of spheres that support H–space structures (nevermind the structure of a topological group or a Lie group). Actually understanding the proof of this is too complicated (probably even by the conclusion of the class), but there are less sharp results that are accessible now: for instance, you could show the Lie group version of the result, or you could learn about the reduction to spheres of dimension of the form $2^n - 1$.

(18) Oftentimes in life there is some background group G that acts on everything in sight, and there is a homotopy theory of G–spaces set up to handle this. It has *many* non-obvious features to learn about.

(19) There's a lot going on in low-dimensional topology and, say, the theory of knots. There are even links to stable homotopy theory: Lipschitz has at least two papers linking Khovanov homology to stable homotopy theory, Hu–Kriz–Kriz have a paper linking this to quantum field theories, ... (These links to stable homotopy theory might be better left for the final paper.)

(20) Quantum field theories, for that matter, are interesting to learn about. The topologists' definition of a topological quantum field theory is a monoidal functor off of a (higher) bordism category. There is a *lot* about these that is known, but I'm not very familiar with it, so I can't really direct you. (The largest homotopy-theoretic paper is Jacob Lurie's solution to the cobordism hypothesis, and before that there were massively important papers of Atiyah and Segal setting up the basics of the theory. Everything in between is a mystery to me, but there's probably a lot to learn, especially if you like physics.)

(21) The Barratt–Priddy–Quillen theorem. There are at least three directions you could take this in: (1) you could learn about algebraic K–theory, (2) you could learn about homological stability, or (3) you could learn about Segal's Γ–spaces.
(22) The h–cobordism theorem ties unstable algebraic topology to manifold geometry.
(23) Recent work of Riehl–Verity (involving a thing called a "cosmos") aims to give an abstract framework for working in $(\infty, 1)$–categories without naming a concrete model. This is interesting and (supposedly) highly accessible.
(24) There are *lots* of models for a model category of spectra: orthogonal spectra, symmetric spectra, ... These all enjoy different categorical properties, which is a strange feature of homotopy theory explained by Lewis's *Is there a convenient category of spectra?*. There are lots of places to read about this, including Cary Malkiewich's pleasant summary, *The stable homotopy category* (just to name one).
(. . .) or anything else you'd like!

2. Final topics

This can should longer, at least 10 pages.
Important dates:

- Topic due: Friday, April 14th
- Rough draft due: Monday, April 24th
- Paper due: Monday, May 1st

(1) The homotopy theory of G–spectra