# Introduction to advanced algebraic topology

References: Switzer's *Algebraic Topology*
Spanier's *Algebraic Topology*

Goal: Introduction to homotopy theory (of spaces).

Category theory

① Decomposition of spaces (co/fiber sequences).

② Invariants constructed from decompositions ($H_*$, $\pi_*$) and their properties (Whitehead, Hurewicz, ...). mod-$\mathcal{C}$ theory, ...

③ Representability theorems (Brown, Adams) and the stable category ($H\mathbb{Z}$, $KU$ and $KO$, $\mathbb{S}$, $MG$, ...).

④ Computation (characteristic classes, Bott periodicity, the Steenrod op^ns, the Adams + Serre SSes, ...)

Here's an example of the kind of analysis you can perform with these tools at your disposal:

Start with your favorite (s.c.) space, like $S^{n\geq 2}$. Its homotopy gp^s are notoriously difficult + important to compute.

② **Thm** (Hurewicz): $H_n(X;\mathbb{Z}) \cong \pi_n X$ for $\pi_{*<n} X = 0$.
So, we get one for free. We also have a decomposition ①

$$X[n+1,\infty) \longrightarrow X \longrightarrow K(\pi_n X, n) \quad ③$$

by a fiber seq^ee s.t. $\pi_* K(\pi_n X, n) = \begin{cases} \pi_n X \text{ if } * = n, \\ 0 \text{ o/w,} \end{cases}$ , $\pi_* X[n+1,\infty) = \begin{cases} \pi_* X \text{ if } * > \\ 0 \text{ o/w.} \end{cases}$

Then,

$$\begin{tikzcd} H_* X \arrow[dr, Rightarrow] & & \\ & \text{Serre spectral sequence ④} \arrow[r, Rightarrow] & H_* X[n+1,\infty). \\ H_* K(\pi_n X, n) \arrow[ur, Rightarrow] & & \end{tikzcd}$$

Go back to ② and repeat...

## Some facts about spaces + their category (Ch. 0)

Today I wanted to remind you of some basic topological facts, so that I can guiltlessly assume them later on.

3 basic constructions:

locality $\rightsquigarrow$ ① For $X = \bigcup A_j$ a decomposition into closed subsets, $\{f : X \longrightarrow Y \text{ cts}\}$ bijects with $\{(f_i : A_i \longrightarrow Y)^{\text{cts}} \mid f_i|_{A_i \cap A_j} = f_j|_{\cdots}\}$.

products $\rightsquigarrow$ ② For $X, Y$ spaces there's a space $X \times Y$ such that $\{f : T \longrightarrow X \times Y^{\text{cts}}\}$ bijects with $\{(f_X : T \longrightarrow X)^{\text{cts}}, f_Y : T \longrightarrow Y)^{\text{cts}}\}$.

quotients $\rightsquigarrow$ ③ For $R$ an equiv$^{\underline{ce}}$ rel$^{\underline{n}}$ on $X$, there is a space $X/R$ s.t. $\{\bar{f} : X/R \longrightarrow Y \text{ cts}\}$ bijects with $\{f : X \longrightarrow Y^{\text{cts}} \mid xRx' \Rightarrow f(x) = f(x')\}$.

<u>Ex</u>: For $A \overset{\neq \emptyset}{\subseteq} X$, define $X/A$ by extending the total rel$^{\underline{n}}$ on $A$ by the identity rel$^{\underline{n}}$ on $X$. As an edge case, set $X/\emptyset = X \sqcup \{*\}$.

<u>Lem</u>: If $Y$ is locally compact, then $\frac{X \times Y}{\alpha \times id} \xrightarrow{\cong} (X/\alpha) \times Y$.

Pf: There is always a cts bijection $\frac{X \times Y}{\alpha \times \beta} \xrightarrow{\varphi} X/\alpha \times Y/\beta$, which we want to show is open. For $\mathcal{U} \subseteq \frac{X \times Y}{\alpha \times id}$ open, and a point $([x_0], y_0) \in \varphi(\mathcal{U})$, pick a nbhd $U_X \times U_Y \subseteq X \times Y$ of $(x_0, y_0)$ with $U_X$ open, $U_Y$ compact w/ $y_0 \in \overset{\circ}{U_Y}$. Set $J = \{x \in X \mid x \times U_Y \subseteq \mathcal{U}\}$; then $J$ is open and $\varphi(J \times \overset{\circ}{U_Y})$ is a nbhd of $([x_0], y_0) \in X/\alpha \times Y$. $\square$

This Lemma is what allows us to "fatten" constructions: for example, setting $I = [0,1]$, $\left(\frac{X}{\alpha}\right) \times I = \frac{X \times I}{\alpha \times 1}$.

It also powers the discussion of <u>relative homotopy</u>:

<u>Lem</u>: If $H : X \times I \longrightarrow Y$ factors as $X \times \{t\} \longrightarrow \frac{X}{\alpha} \times \{t\} \longrightarrow Y$ $\forall t$, then it factors as a whole as $X \times I \longrightarrow \frac{X}{\alpha} \times I \xrightarrow{\tilde{H}} Y$. $\square$

<u>Lem</u>: If $A \subseteq X$ is closed and $H(a, t) = H(a', t)$ $\forall a, a' \in A$ + $t \in I$, then it factors as $X \times I \longrightarrow \frac{X}{A} \times I \longrightarrow Y$. $\square$

Function spaces/
exponential objects

④ For $X, Y$ spaces, $Y^X = \{f : X \to Y \text{ cts}\}$. If $X$ is locally compact, then $Y^X \times X \xrightarrow{ev} Y$ is cts. If $X$ and $Z$ are additionally Hausdorff, then $Y^{Z \times X} \longrightarrow (Y^Z)^X$ is a homeo$^{\underline{m}}$.

We will perpetually arrange to be in this situation, so that we have access to these function spaces.

We will also often want to track a preferred point in a space: $(X, x_0)$, and restriction attention to maps $f : (X, x_0) \xrightarrow{f} (Y, y_0)$ that are cts functions $f : X \to Y$ s.t. $f(x_0) = y_0$. (This is Spaces$_{*/}$.)
More generally, we might want preferred subspaces:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ A \arrow[u, phantom, sloped, "\subset"] \arrow[r, dashed, "f|A"] & B \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}$$

One can re-envision the op$^{\underline{ns}}$ ①-④ for such "relative" objects.
For instance, $(X, A) \times (Y, B) = (X \times Y, (X \times B) \cup (A \times Y))$, or

$$(Y, B)^{(X, A)} = \{f : X \to Y \text{ cts} \mid f(A) \subseteq B\}.$$

These interact as expected:

$$(Y, B)^{(Z, C) \times (X, A)} = \left((Y, B)^{(Z, C)}\right)^{(X, A)}.$$

In particular,

$$\left((Y, y_0)^{(Z, z_0)}\right)^{(X, x_0)} = (Y, y_0)^{(X \times Z,\ \underbrace{(X \times z_0) \cup (x_0 \times Z)}_{\shortparallel})}$$

product in relative spaces

$$X \vee Z, \text{ and } \frac{X \times Z}{X \vee Z} =: X \wedge Z.$$

product in Spaces$_{*/}$

coproduct in Spaces$_{*/}$

"monoidal product" in Spaces$_{*/}$

Perspective on the fundamental group (2 - 2.23)

Recall that the pathspace of $X$ is $X^I$, $I = [0,1]$.

Def: $\pi_0(X)$ is the set of path-components of $X$, i.e., $[x] = [x']$ when $\exists\, \gamma \in X^I$ s.t. $\gamma(0) = x$, $\gamma(1) = x'$.

Writing $[Y, X]$ for the set of homotopy classes of f$^{\text{ns}}$ $Y \to X$, we also have $\pi_0 X = [*, X]$, or $\pi_0(X, x_0) = [S^0, X]$ for $S^0 = (\{\pm 1\}, 1)$. In terms of the exponential objects from last time, homotopy classes themselves can be defined as $\pi_0 X^Y$, from which it follows that $[Z \wedge X, Y] = [X, Y^Z]$ for ptd spaces.

Let's use this to perturb the definition of a fundamental group:

$\pi_1(X) := \{$homotopy classes of pointed loops in $X\}$

~~$= \pi_0 X^{(S^1)}$ = [illegible]~~ $= [S^1, X] = [S^0 \wedge S^1, X] = [S^0, X^{(S^1)}]$

$= \pi_0 X^{(S^1)}$.

One might wonder what properties of $Y + X$ make $[$~~[illegible]~~$Y, X]$ into a gp, since we know that $[S^1, -]$ and $[S^0, (-)^{S^1}]$ are gp-valued.

Recall: A group is a set $G$ w/ maps $\mu: G \times G \to G$, $e: * \to G$, $\chi: G \to G$ satisfying

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu \times 1"] \arrow[d, "1 \times \mu"] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G \end{tikzcd}, \quad \begin{tikzcd} G \arrow[r, "e \times 1"] \arrow[dr, equal] & G \times G \arrow[d, "\mu"] & G \arrow[l, "1 \times e"'] \arrow[dl, equal] \\ & G & \end{tikzcd}, \quad \begin{tikzcd} G \arrow[r, "\chi \times 1"] \arrow[d, "0"] & G \times G \arrow[d, "\mu"] & G \arrow[l, "1 \times \chi"'] \arrow[d, "0"] \\ * \arrow[r, "e"] & G & * \arrow[l, "e"'] \end{tikzcd}.$$

This definition is categorical, using diagrams and products. Recall the product of spaces from the last lecture; its defining property might be better written as $\text{Spaces}(T, X \times Y) \cong \text{Spaces}(T, X) \times \text{Spaces}(T, Y)$ —— or $\text{Spaces}(T, -)$ converts products to products.

Def: An H-space $K$ is a space w/ maps $\mu, e, \chi$ making the group diagrams commute up to homotopy.

Cor: The functor $[-, K]$ is valued in groups. □

$(X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} K \times K \xrightarrow{\mu} K.)$

Previously, you defined an H-space structure on $X^{(S^1)} =: \Omega X$: two loops can be scaled + concatenated, and loops can be run backward. Hence, not only is $\pi_0 \Omega X = [{*}S^0, \Omega X]$ a group, but $[Y, \Omega X]$ always is.

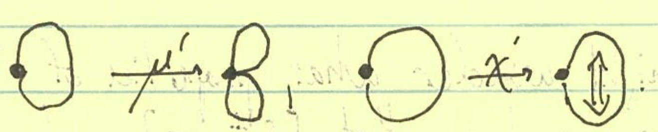
What about the other formulation? We also have $\pi_1 X = [S^1, X]$, and the magic may not rest in the output "$\Omega X$" but in $S^1$ alone.

<u>Def</u>: An H-cogroup $K$ has $\mu' : K \to K \vee K$, $\chi' : K \to K$ such that

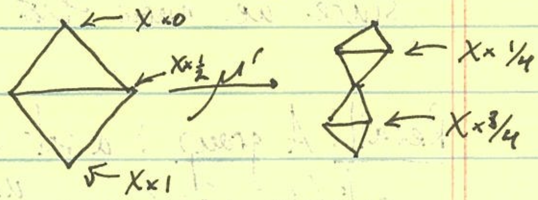
$$\begin{tikzcd} K\vee K\vee K & K\vee K \arrow[l, "\mu'\vee 1"'] \\ K\vee K \arrow[u, "1\vee\mu'"] & K \arrow[l, "\mu'"] \arrow[u, "\mu'"'] \end{tikzcd}, \quad \begin{tikzcd} K & K\vee K \arrow[l, "\text{left}"'] \arrow[r, "\text{right}"] & K \\ & K \arrow[u, "\mu'"] \arrow[ul, equal] \arrow[ur, equal] & \end{tikzcd}, \quad \begin{tikzcd} K & K\vee K \arrow[l, "\text{fold}\circ(1\vee\chi')"'] \arrow[r, "\text{fold}\circ(\chi'\vee 1)"] & K \\ & K \arrow[u, "\mu'"] \arrow[ul, "0"] \arrow[ur, "0"'] & \end{tikzcd}.$$

Again, $\mathrm{Spaces}_*(X \vee Y, T) = \mathrm{Spaces}_*(X, T) \times \mathrm{Spaces}(Y, T)$, so $\mathrm{Spaces}(K, -)$ is naturally group-valued.

<u>Ex</u>: $S^1$ is an H-cogroup.



<u>Ex</u>: In fact, $\underbrace{S^1 \wedge X}_{\overset{\shortparallel}{\Sigma X}}$ is an H-cogroup for <u>any</u> $X$.



<u>Lem</u>: The adjunction $[\Sigma X, Y] \cong [X, \Omega Y]$ is an iso<sup></sup>$^{\underline{m}}$ of gp$^{\underline{s}}$.

Pf: This is a matter of writing out the formulas for

$$\Sigma X \xrightarrow{\mu'} \Sigma X \vee \Sigma X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\Delta'} Y \quad \text{and}$$

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} \Omega Y \times \Omega Y \xrightarrow{\mu} Y. \qquad \square$$

Higher homotopy groups

Higher homotopy groups have a similar bunch of def$^{\underline{ns}}$:

Def: $\pi_n X = [\Sigma^n(S^0), X] = [\Sigma^{n-1}(S^0), \Omega X] = \cdots = [S^0, \Omega^n X]$.

In the middle stages, there are two multiplications on the homotopy mapping sets coming from $\Sigma$ and $\Omega$ both.

Lem (Eckmann–Hilton): Let $S$ be a set w/ two products $\circ$ + $*$ that share a unit + $(x * x')\circ(y * y') = (x \circ y) * (x' \circ y')$. Then $\circ = *$, and both are assoc. + comm..

Pf: $x\circ y = (x*e)\circ(e*y) = (x\circ e)*(e\circ y) = x*y$.

$x\circ y = (e*x)\circ(y*e) = (e\circ y)*(x\circ e) = y*x$. □

Cor: $[\Sigma^{n-1}(S^0), \Omega X]$ has only one multiplication + it is comm.

Pf: Consider $[K, L]$ for $K$ an H-cogp and $L$ an H-gp. We want

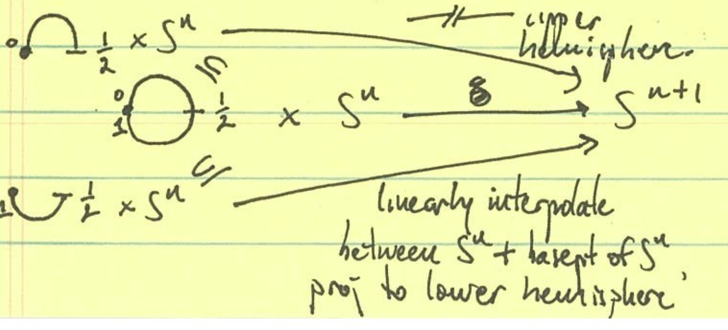
$$\begin{tikzcd}
 & K\times K \arrow[r] \arrow[dr, phantom, "\#"] & (K\vee K)\times(K\vee K) \arrow[r] & (L\vee L)\times(L\vee L) \arrow[r] & L\times L \arrow[d] \\
K \arrow[ur] \arrow[r] \arrow[dr] & \Delta K \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] \arrow[d, phantom, "\#"] & \begin{matrix}(K\times *)\times(K\times *)\\ \cup \\ (*\times K)\times(*\times K)\end{matrix} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] \arrow[r, phantom, "\#", shift left=4] \arrow[d, "|\times T\times|"'] \arrow[dr, phantom, "\#"] & \begin{matrix}(L\times *)\times(L\times *)\\ \cup \\ (*\times L)\times(*\times L)\end{matrix} \arrow[u, phantom, sloped, "\subseteq"] \arrow[ur] \arrow[ur, phantom, "\#", shift left=3] \arrow[d, "|\times T\times|"'] \arrow[dr, phantom, "\#"] & L \\
 & K\vee K \arrow[r] & (K\times K)\vee(K\times K) \arrow[r] & (L\times L)\vee(L\times L) \arrow[r] & L\vee L \arrow[u]
\end{tikzcd}$$

to commute. ① ② ③

$$\left[\begin{matrix}(L\times L)\vee(*\times *)\\ \cup \\ (*\times *)\vee(L\times L)\end{matrix}\right]$$

□

Lem: For all $n \geq 0$, $S^1 \wedge S^n \cong S^{n+1}$ a homeo$^{\underline{m}}$.



□ $\Longrightarrow$ $\pi_n X = [S^n, X]$ is an ab. gp. for $n \geq 2$.

## Exact sequences in Spaces

Recall that $N \xrightarrow{f} G \xrightarrow{g} H$ is called <u>exact</u> when $\ker g = \operatorname{im} f$. Last time, we got a lot of mileage out of putting structure on spaces such that $[K, -]$ or $[-, L]$ is valued in groups. Today we are after something similar: when do continuous maps $A \longrightarrow B \longrightarrow C$ induce "exact seq^ces" $\underbrace{[-,A] \to [-,B] \to [-,C]}_{\text{exact}}$, or $\underbrace{[A,-] \leftarrow [B,-] \leftarrow [C,-]}_{\text{coexact}}$?

<u>Def</u>: A seq^ce of pointed sets $X \xrightarrow{f} Y \xrightarrow{g} Z$ is <u>exact</u> if $\operatorname{im} f = g^{-1}(*)$.

<u>Lem</u>: Any map $X \xrightarrow{f} Y$ extends to a coexact seq^ce $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Pf: This starts with a construction: set $CX = X \wedge I$ and $Z = Y \cup_f CX = \frac{Y \amalg CX}{f(x) \sim (x,1)}$. Now consider $[Y \cup_f CX, T] \to [Y, T] \to [X, T]$, and a $f^{\underline{n}}$ $\varphi: Y \longrightarrow T$. If $\varphi \circ f$ is null, then a nullhomotopy defines a map $CX \longrightarrow T$ agreeing with $f$ on the edge of $I$, and vice versa. □

We can iterate this process: $X \to Y \xrightarrow{g} Y \cup_f CX \to (Y \cup_f CX) \cup_g CY \to \cdots$.

These spaces start to look gross quickly, but actually they are quite nice.

<u>Lem</u>: $(Y \cup_f CX) \cup_g CY \longrightarrow ((Y \cup_f CX) \cup_g CY)/CY$ is a h^tpy equiv^ce. □

<u>Lem</u>: For $A \subseteq X$ a subspace, $(X \cup_i CA)/CA \cong X/A$. □

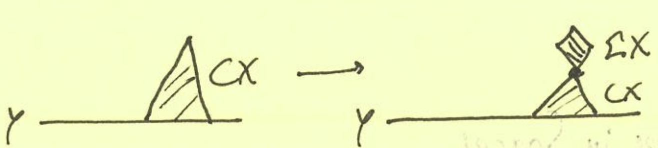
<u>Cor</u>: $(Y \cup_f CX) \cup_g CY \simeq ((Y \cup_f CX) \cup_g CY)/CY \cong (Y \cup_f CX)/Y \cong CX/X \cong \Sigma X$, and $((Y \cup_f CX) \cup_g CY) \cup_{g'} C(Y \cup_f CX) \simeq \Sigma Y$, hence the coexact seq^ce takes the form $X \xrightarrow{f} Y \longrightarrow Y \cup_f CX \longrightarrow \Sigma X \xrightarrow{-f} \Sigma Y \longrightarrow \Sigma(Y \cup_f CX) \longrightarrow \cdots$

cf 2.39.

Coupling this to our results from last time, we have an exact seq^ce

$$\underbrace{[X,T] \leftarrow [Y,T] \leftarrow [\overbrace{Y \cup_f CX}^{Z},T]}_{\text{sets}} \leftarrow \underbrace{[\Sigma X,T] \leftarrow [\Sigma Y,T] \leftarrow [\Sigma Z,T]}_{\text{groups}} \leftarrow \underbrace{[\Sigma^2 X,T] \leftarrow [\Sigma^2 Y,T] \leftarrow \cdots}_{\text{abelian gps.}}$$

"Exact seq^ces" are best behaved on abelian gp^s. What about the edge cases?



Contr: There is a map $Y \cup_f CX \longrightarrow (Y \cup_f CX) \vee \Sigma X$, which on $[-,T]$ gives an action $[Z,T] \times [\Sigma X, T] \longrightarrow [Z,T]$.

Lem: 
$$\begin{tikzcd}[Z,T] \arrow[rr] \arrow[dr] & & {[Y,T]} \\ & {[Z,T]/\text{action of }[\Sigma X,T]} \arrow[ur] & \end{tikzcd}$$
.

Pf: Certainly there is a factorization; we need injection. If two maps $Z \xrightarrow{f_1, f_2} T$ restrict to the same map on $Y$, restrict them to $CX$ to get a map
$$\begin{tikzcd}CX_{top} \arrow[drr, bend left, "f_1"] & & \\ & \subseteq \Sigma X \arrow[r, "d(f_1,f_2)"] & T \\ CX_{bottom} \arrow[ur, phantom, "\subseteq"] \arrow[urr, bend right, "-f_2"] & & \end{tikzcd}$$
. Then $f_1 = d(f_1,f_2) \cdot f_2$. □

There are also dual results for exact seq$\underline{^{ces}}$ of spaces: note that a null-homotopy of a map $X \longrightarrow Y \equiv$ a map $X \times I = CX \rightarrow Y \equiv$ a map $X \rightarrow PY = Y^I$.

Lem: Any map $X \xrightarrow{f} Y$ extends to an exact seq$\underline{^{ce}}$ $P_f \longrightarrow X \longrightarrow Y$, where $P_f = \{(x,\gamma) \in X \times PY \mid f(x) = \gamma(1)\}$. □

Lem: Iterating this gives $\cdots \rightarrow \Omega X \longrightarrow \Omega Y \longrightarrow P_f \longrightarrow X \xrightarrow{f} Y$. □

In particular, applying $\pi_0$ and using the def$\underline{^{n}}$ $\pi_n X = \pi_0(\Omega^n X)$, we get an exact seq$\underline{^{ce}}$ of homotopy groups:
$$\cdots \rightarrow \pi_2 X \longrightarrow \pi_2 Y \longrightarrow \pi_1 P_f \longrightarrow \pi_1 X \longrightarrow \pi_1 Y \longrightarrow \pi_0 P_f \longrightarrow \pi_0 X \longrightarrow \pi_0 Y.$$

One of our goals in this class will be to understand what $\pi_*$ earns us, how it changes as $X$ changes, and how $X$ can be effectively directed to control the behavior of $\pi_*$ along the way.

Relative homotopy groups

The construction $P_f$ from the previous lecture is a little mysterious. We will explore it in two lights, today + in   lectures.

For the time being, we restrict attention to inclusions $i : A \to X$, and we think of this data as an object $(X, A, x_0)$ in a category of pairs of topological spaces. Then $P_i = (X, A, x_0)^{(I, \partial I, 0)}$, and we make the following adjunction juggle:

Def: $\pi_{n-1}(P_i; x_0) = [S^{n-1}, *; P_i, x_0] = [(D^n, S^{n-1}), (X, A)] =: \pi_n(X, A)$.

That is, $\pi_n(X,A)$ consists of $n$-disk maps into $X$ with $\partial D^n$ lying in $A$.

Cor: There is a lexseq:

$$\cdots \to \pi_{n+1}(X,A) \to \pi_n A \to \pi_n X \to \pi_n(X,A) \to \pi_{n-1} A \to \cdots . \ \square$$

Rem: $\pi_n(X; x_0) = \pi_n(X, \{x_0\})$. So, we can regard the map $\pi_n X \to \pi_n(X,A)$ as induced by $(X, \{x_0\}) \hookrightarrow (X, A)$.

One of the uses of these relative gps is to study the discrepancy between $\pi_* A$ and $\pi_* X$. This motivates a definition:

Def: A pair $(X,A)$ is n-connected if $\pi_{\leq n}(X,A) = 0$. Equiv$^{tly}$, the map $\pi_* A \to \pi_* X$ is an iso for $* < n$ + onto for $* = n$.

Def: $A \subseteq X$ is a weak equivalence if it is $\infty$-connected.

Rem: We can extend these definitions to a generic $f : Y \to X$ using the mapping cylinder construction:

$$\begin{tikzcd}
Y \times 1 \arrow[dr, hook] \arrow[drr, hook] & & M_f \arrow[d, equal] & \\
& Y \times I \arrow[r] & (Y \times I) \cup_{f \times 0} X & X \arrow[l, "\simeq"'] \\
Y \times 0 \arrow[ur, hook] \arrow[urr] \arrow[urrr, bend right] & & &
\end{tikzcd}$$

.

There is another tool used to interrelate "genuinely" relative gp$^s$ amongst each other. Let $x_0 \in B \subseteq A \subseteq X$ be a pair of inclusions, then

$$\begin{tikzcd}
\cdots & \pi_{n+1}(X,A) \arrow[rr] \arrow[dr] & & \pi_n(A,B) \arrow[rr] \arrow[dr] & & \pi_{n-1}B \arrow[rr] \arrow[dr] & & \pi_{n-1}X & \cdots \\
\cdots & & \pi_n A \arrow[ur] \arrow[dr] & & \pi_n(X,B) \arrow[ur] \arrow[dr] & & \pi_{n-1}A \arrow[ur] \arrow[dr] & & \cdots \\
\cdots & \pi_n B \arrow[rr] \arrow[ur] & & \pi_n X \arrow[rr] \arrow[ur] & & \pi_n(X,A) \arrow[rr] \arrow[ur] & & \pi_{n-1}(A,B) & \cdots
\end{tikzcd}$$

lift $y-y'$. $x \mapsto 0$, $y \mapsto x$, $x \mapsto 0$, $y'$, $w \mapsto z$, $y \mapsto z$, $z \mapsto 0$

commutes. Tracing through the middle, we see an interesting seq$^{ce}$.

Lem: This seq$^{ce}$ is exact.

Pf sketch: Checking that the composites are zero is easy enough. The most confusing ker = im condition is the first one pictured, and we draw the diagram chase in green. □

$\boxed{\text{This all smacks of homology.}}$

As an application, consider a retraction $A \underset{i}{\overset{r}{\leftrightarrows}} X$ with $r \circ i \simeq id_A$.

Then: $\pi_* A \hookrightarrow \pi_* X$ is an inclusion.

$\Longrightarrow$ the boundary map $\pi_*(X,A) \xrightarrow{\partial} \pi_{*-1} A$ is zero.

$\Longrightarrow$ there are ses$^s$ $\begin{tikzcd} 0 \arrow[r] & \pi_n A \arrow[r, hook] & \pi_n X \arrow[r] \arrow[l, bend left] & \pi_n(X,A) \arrow[r] & 0 \end{tikzcd}$.

For $n \geq \cancel{2}3$, all the gps are abelian and hence $\pi_n X \cong \pi_n A \oplus \pi_n(X,A)$.

Below, $\pi_2 X$ is a semidirect product of $\pi_2 A$ and $\pi_2(X,A)$.

## The action of $\pi_1$

Before continuing our study of $P_f$, there is one other "old" fact about homotopy gp$^{s}$ we should investigate: their dependence on $x_0 \in X$. You might even remember that paths $\gamma$ in $X$ induce iso$^{ms}$ $\pi_1(X, \gamma(0)) \to \pi_1(X, \gamma(1$

This fits into a framework we have considered already. We begin with the algebraic thing we are trying to model.

**Def:** For $G, A$ gp$^{s}$, an action $\alpha : G \times A \longrightarrow A$ is compatible when $g(a_1 a_2) = (g a_1)(g a_2)$, i.e.,

$$\begin{tikzcd}
G \times A \times A \arrow[r, "\Delta \times 1 \times 1"] \arrow[d, "1 \times \mu"] & G \times \overline{G \times A} \times A \arrow[r, "\cong"] & G \times \overline{A \times G} \times A \arrow[r, "\alpha \times \alpha"] & A \times A \arrow[d, "\mu"] \\
G \times A \arrow[rrr, "\alpha"] & & & A
\end{tikzcd}$$

**Ex:** $G$ acts compatibly on itself by conjugation.

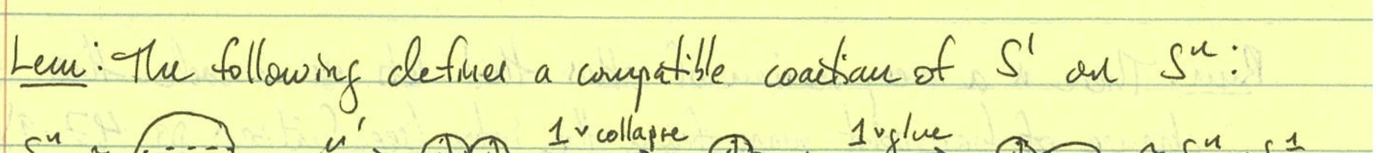
**Ex:** For $A$ abelian (i.e., a $\mathbb{Z}$-module), this is equiv$^{t}$ to a $\mathbb{Z}[G]$-module st$^{r}$.

The theorem we want to prove is that $\pi_1 X$ acts compatibly on $\pi_n X$, $n \geq 1$.

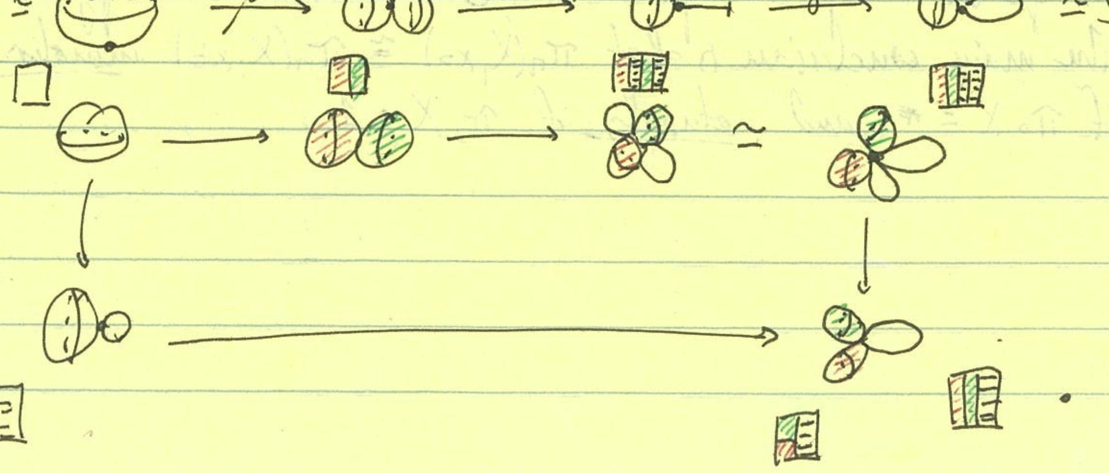
**Def:** An H-cogroup $K$ acts compatibly on an H-cogroup $L$ when the diagram dual to the one above commutes.

**Cor:** For such H-cogroups, the action of $[K, T]$ on $[\overset{L}{\cancel{K}}, \overset{T}{\cancel{L}}]$ is compatible. □

**Lem:** The following defines a compatible coaction of $S^1$ on $S^n$:



Pf: □



(The proof is a little more obnoxious when $n=1$.)

Rem: There is also a relative version of this story: $S^1$ has a compatible coaction on $(CS^n, S^n)$, hence $\pi_1 A$ acts compatibly on $\pi_n(X,A)$ for $n \geq 2$.

Lem: The action satisfies various naturalities:

i) $(X,A) \to (Y,B)$ induces

$$\begin{tikzcd} \pi_1 X \times \pi_n X \arrow[r] \arrow[d] & \pi_n X \arrow[d] \\ \pi_1 Y \times \pi_n Y \arrow[r] & \pi_n Y \end{tikzcd} + \begin{tikzcd} \pi_1 A \times \pi_n(X,A) \arrow[r] \arrow[d] & \pi_n(X,A \arrow[d] \\ \pi_1 B \times \pi_n(Y,B) \arrow[r] & \pi_n(Y,B) \end{tikzcd}$$

ii)

$$\begin{tikzcd} \pi_1 A \times \pi_n A \arrow[r] \arrow[d] & \pi_n A \arrow[d] \\ \pi_1 X \times \pi_n X \arrow[r] & \pi_n X \end{tikzcd}$$

commutes, as does

$$\begin{tikzcd} \pi_1 A \times \pi_n(X,A) \arrow[r] \arrow[d] & \pi_n(X,A) \arrow[d] \\ \pi_1 A \times \pi_{n-1} A \arrow[r] & \pi_{n-1} A \end{tikzcd}.$$

iii) The action of $\pi_1 X$ on itself is by conjugation: $\gamma \cdot \alpha = \gamma \alpha \gamma^{-1}$. □

Rem: A weak equiv^ce^ $f: Y \to X$ thus induces an iso^m^ of $\mathbb{Z}[\pi_1 X]$-modules, in addition to just abelian gp^s^; a strictly stronger condition. There is more such structure; this is the start of the study of <u>Π-algebras</u>.

Rem: There is a messy version of this that lets us encode the change of basepoint maps from the intro (see Switzer pg. 47-9). The main conclusion is that $\pi_n(X,x_0) \cong \pi_n(X,x_0')$ <u>unnaturally</u> if $\pi_0 X = *$, and <u>naturally</u> if $\pi_1 X = 1$.

## Fibrations (4 - 4.8)

We now have a very explicit model for the spaces in a coexact sequence, via cones + suspensions. We would like to generate a similarly nice description of exact sequences.

Ex: There's an iso$^{\underline{m}}$ $\pi_n(X\times Y)\cong\pi_n(X)\times\pi_n(Y)$ for $n\geq 0$.
Rearranged to be an exact sequence, the maps $Y \rightleftarrows X\times Y \rightleftarrows X$ induce a split-exact sequence $\cdots\to\pi_n Y\rightleftarrows\pi_n(X\times Y)\rightleftarrows\pi_n X\to\cdots$.
We would like to ~~generalize~~ axiomatize the part of the geometry of the Cartesian product that we need to induce (non-split) exact seq$^{\underline{ces}}$ on $\pi_*$.

Def: $E\xrightarrow{p}B$ has the homotopy lifting property w/r/t a space $X$ when

$$\forall\quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "-\times 0"'] & E \arrow[d, "p"] \\ X\times I \arrow[r, "H"] & B \end{tikzcd} \quad \overset{\exists}{\rightsquigarrow} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "-\times 0"'] & E \arrow[d, "p"] \\ X\times I \arrow[r, "H"] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd},$$

i.e., homotopies in B lift to homotopies in E.

A fibration has the HLP for all spaces. A weak fibration has the HLP for at least the disks $D^n$. The fiber is $p^{-1}(b_0)\subseteq E$.

Ex: $X\times Y\to X$ works. ← fib. Ex: $PX\to X$ works, w/ fiber $\Omega X$.
(Rem: $PX\simeq *$.)

Lem: Consider $p:E\to B$, $B'\subseteq B$, $E':=p^{-1}(B')\subseteq E$. If $p$ has the HLP for $X\times I$, then $p'$ has the HLP for $X$!

$p':(E,E')^{(I,\partial I)}\longrightarrow(B,B')^{(I,\partial I)}$

$$\text{Pf: }\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & (E,E')^{(I,\partial I)} \arrow[d, "p'"] \\ X\times I \arrow[r, "H"] & (B,B')^{(I,\partial I)} \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} X\times(I\vee I) \arrow[r, "f''"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X\times I\times I \arrow[r, "H'"] & B \end{tikzcd}.$$

There is a homeo$^{\underline{m}}$ $I \vee I \simeq I$ extending to a homeo$^{\underline{m}}$ $I \times I \overset{h'}{\cong} I \times I$.

Use this to juggle:

$$\begin{tikzcd}
X \times I \arrow[r, "f'' \circ h^{-1}"] \arrow[d, "- \times 0"'] & E \arrow[d, "p"] \\
X \times I \times I \arrow[r, "H' \circ h^{-1}"] \arrow[ur, dashed, "\exists \widetilde{H}'"] & B
\end{tikzcd}$$

The HLP gives $\widetilde{H}'$.

One checks that de-juggling this gives a filler $\widetilde{H}$. □

<u>Cor</u>: If $p$ as above is a weak fib$^{\underline{n}}$, then $\pi_n(E, E') \xrightarrow{\cong} \pi_n(B, B')$ $\forall n \geq 1$.

Pf: Given $\omega: (I, \partial I) \to (B, B')$, consider just

$$\begin{tikzcd}
\ast \arrow[r, "e_0"] \arrow[d, hook] & E \arrow[d, "p"] \\
\ast \times I \arrow[r, "\omega"] \arrow[ur, dashed, "\widetilde{\omega}"] & B
\end{tikzcd}$$

Since $\widetilde{\omega}(1) \in B'$ and $E' := p^{-1}(B')$, this gives a map $\widetilde{\omega}: (I, \partial I) \to (E, E')$ and $\pi_1(E, E') \twoheadrightarrow \pi_1(B, B')$.

If $\omega_1, \omega_2: (I, \partial I) \to E$ are two rel$^{\underline{ve}}$ h$^{\underline{ty}}$ classes and $H: I \times I \to B$ is a homotopy connecting them in $B$, take

$$\begin{tikzcd}
\text{"}\sqcup\text{"} \arrow[r, "\omega_1 \sqcup \bullet \sqcup \omega_2"] \arrow[d, hook] & E \arrow[d] \\
I \times I \arrow[r, "H"] \arrow[ur, dashed, "\widetilde{H}"] & B
\end{tikzcd}$$

and use the trick from the previous pf to produce $\widetilde{H}$. Thus $\pi_1(E, E') \hookrightarrow \pi_1(B, B')$ as well.

To get $\pi_n$, $n > 1$, use relative pathspaces to induct up. □

Pf.

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n F \arrow[r] \arrow[d, equal] & \pi_n E \arrow[r] \arrow[d, equal] & \pi_n(E, F) \arrow[r] \arrow[d, "\cong"] & \pi_{n-1} F \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \pi_n F \arrow[r] & \pi_n E \arrow[r] & \pi_n B = \pi_n(B, b_0) \arrow[r] & \pi_{n-1} F \arrow[r] & \cdots
\end{tikzcd}$$

<u>Cor</u>: $\cdots \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F \to \cdots$ is a long exact sequence. □

<u>Cor</u>: If $E$ is contractible, then $\pi_n B \cong \pi_{n-1} F$. □

## Fiber bundles and examples (4.9 –)

It turns out that fibrations are all over.

**Def**: A *fiber bundle* is data $F \subseteq E \xrightarrow{p} B$ such that $B$ has an open cover $\{U_\alpha\}_\alpha$ and local homeo^ms $\varphi_\alpha : U_\alpha \times F \longrightarrow p^{-1}(U_\alpha)$ with $p\varphi_\alpha = p_u$

**Lem**: Every fiber bundle is a weak fibration.

Pf: Fix an open cover and subdivide $D^n \times I$ so finely that you only ever live in one piece of the cover, then work locally. □

**Rem**: If $B$ is paracompact, then $p$ is actually a fibration.

**Def**: If $H \leq G$ is a closed subgp of a top. gp and $H \in G/H$ has an open nbhd $U$ with a section $\underbrace{U \xrightarrow{s} G \xrightarrow{f}}_{} G/H$, then $p$ is a fiber bundle w/ fiber $H$.

Pf: By left-mult. by $g \in G$, the condition at $H$ gives section near all $gH \in G/H$. The sections become the data of a fiber bundle by $U_{gH} \times H \xrightarrow{m\circ(s_g \times 1)} G$. □

Stiefel Manifolds **Ex**: $O(n) \hookrightarrow O(n+k)$ by block matrices $\left[\begin{array}{c|c} O(n) & 0 \\ \hline 0 & I \end{array}\right]$. The quotient is the space of orthonormal $k$-frames in $\mathbb{R}^{n+k}$. (For example, $O(n)/O(n-1)$ is homeo^c to $S^{n-1}$.) To construct a local section, note there is an open nbhd$^U$ of $(e_{n+1}, \dots, e_{n+k})$ determined by $\det(e_1|\cdots|e_n|u_{n+1}|\cdots|u_{n+k}) \neq 0$. On $U$, the local section is defined by Gram-Schmidt.

Grassmannian **Ex**: The further quotient of $O(n+k)/O(n)$ by $O(k) = \{\left[\begin{array}{c|c} I & 0 \\ \hline 0 & O(k) \end{array}\right]\}$ is the space of $k$-dim^l subspaces in $\mathbb{R}^{n+k}$. This quotient also has a local section: an open nbhd of $\langle e_{n+1}, \dots, e_{n+k} \rangle$ is those subspaces $W$ with $W \cap \langle e_1, \dots, e_n \rangle = 0$. The projection of $e_{n+1}, \dots, e_{n+k}$ into $W$ ~~give~~ + Gram-Schmidt give orthonormal $k$-frames.

Ex: $SO(n) \to O(n) \xrightarrow{\det} O(1)$ w/ cross-section $\pm 1 \mapsto \left(\begin{array}{c|c} \pm 1 & 0 \\ \hline 0 & I \end{array}\right)$.

Ex: All the same linear algebra with $U(n)$ and $\mathbb{C}^n$.
or with $Sp(n)$ and $\mathbb{H}^n$.

Ex: $\pi_m SO(n) \cong \pi_m O(n)$ for $m \geq 1$.

Def: A *covering* of $B$ is a fiber bundle with discrete fiber.

Ex: $\mathbb{Z} \hookrightarrow \mathbb{R} \to S^1$, local section $\log(-) \cdot \frac{1}{2\pi i}$.
$\Longrightarrow \pi_1 S^1 \cong \mathbb{Z}$ and $\pi_{* \neq 1} S^1 = 0$.

Lem: Discrete $G$ acts on $X$ *properly discontinuously* when (a) $\forall x \, \exists U_x \ni x$ with $g \cdot U_x \cap U_x \neq \emptyset \Rightarrow g = e$ and (b) $\forall x, y$ in different orbits $\exists U_x \ni x, U_y \ni y$ with $g U_x \cap U_y = \emptyset \; \forall g$. In this case, $G \to X \xrightarrow{p} X/G$ is a covering (and $X/G$ is Hausdorff).

Pf: For $[x] \in X/G$, choose a nbhd $U_x \ni x$ guaranteed by (a), and let $p(U_x)$ be the nbhd of $[x]$ in the base. We define $p^{-1}(p(U_x)) \longrightarrow G \times p(U_x)$ by the unicity in (a). □

Rem: i) If $\pi_0 X = 0$, then $0 \to \pi_1 X \to \pi_1 X/G \to \pi_0 G \to 0$ actually a gp hom.
If $\pi_1 X = 0$, then $\pi_1 X/G \cong \pi_0 G$, hence we get a $G$-action on $\pi_n(X/G) \; \forall n \geq 2$. In fact, $\pi_n X \to \pi_n(X/G)$ respects this action.

ii) A covering is *regular* if $p_*(\pi_1 E) \subseteq \pi_1 B$ is normal.
All regular covers arise as quotient by a $G$-action.

iii) $G$ finite, $X$ Hausdorff, $G \circlearrowright X$ has no fixed pts
$\Longrightarrow$ the action is properly discontinuous.

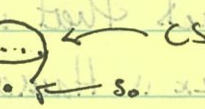
iv) $\mathbb{Z} \circlearrowright \mathbb{R}$ by $1 \cdot x = (x+1)$ is properly discontinuous.

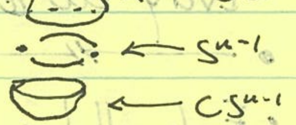
## CW-complexes (5)

A CW-structure on a space $X$ is a presentation of $X$ by inductively "attaching $n$-cells."

**Def:** Given a space $Y$, ~~we say that~~ and a continuous map $g : \vee_\alpha S^{n-1}_\alpha \to Y$, we say that $Y \cup_g \vee_\alpha CS^{n-1}$ is formed from $Y$ by attaching $n$-cells.

**Def:** Let $X^{-\infty} = \cdots = X^{-1} = \{x_0\}$, and let $X^n$ be formed from $X^{n-1}$ by attaching $n$-cells. The union $X$ of the $X^n$ (with the weak topology) is called a CW-structure on $X$. (Or, if $A$ is a CW-complex, then setting $X^{-\infty} = \cdots = X^{-1} = A$ gives rise to a relative CW-structure $(X, A)$.)

**Ex:**
- $S^n$ with cell structure 

- $S^n$ with an inductive cell structure: 

- $S^\infty = \varinjlim S^n$.
- $\mathbb{RP}^n, \mathbb{CP}^n, \mathbb{HP}^n$ constructed inductively by $P^n = A^n \cup P^{n-1}$.
- $\mathbb{RP}^\infty, \mathbb{CP}^\infty, \mathbb{HP}^\infty$, their colimits.

The inductive presentation means we can attack problems cell-by-cell. For instance:

**Cor:** A fn $f : X \to Y$ is cts iff $\vee_n \vee_{\alpha \in A_n} CS^{n-1}_\alpha \to X \xrightarrow{f} Y$ is continuous. $\square$

**Lem:** If $X$ is CW and $Y$ is finite CW, then $X \times Y$ is CW using the homeo $D^n \times D^m \cong D^{n+m}$ $\square$

**Cor:** If $X$ is CW and $A \subseteq X$ is a subcomplex, then $X/A$ is CW. $\square$

**Cor:** Homotopies can also be constructed inductively. $\square$

Rem: If $X$ and $Y$ are infinite, then the weak topology on $X \times Y$ may not agree with the product of weak topologies. We mostly concede this point and pick the one we want (usually the former).

Ex: $S^n \longrightarrow Y$ agree with maps $D^n \longrightarrow Y$ sending $\partial D^n$ to $y_0$.

Ex: Given $S^{n-1} \longrightarrow Y$ and two choices of null-homotopies

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, phantom, "\cap"] & Y \\ CS^{n-1} \arrow[ur, shift left, "H_1"] \arrow[ur, shift right, "H_2"'] & \end{tikzcd}$$

, we can form a difference class $S^n \longrightarrow Y$.

Ex: Given $S^1 \xrightarrow{\omega} Y$ and a null-homotopy

$$\begin{tikzcd} S^1 \arrow[r, "2\omega"] \arrow[d, phantom, "\cap"] & Y \\ D^2 \arrow[ur, "H"] & \end{tikzcd}$$

, we can form $\mathbb{RP}^2 \longrightarrow Y$.

Some topological facts that might interest you:

- Every CW-complex is Hausdorff.
- —— 〃 —— is the disjoint union of the interiors of its cells.
- Each cell has only finitely many immediate faces.
- More generally, any compact subset has this property.

Lem: Let $X = A \cup_{\cancel{f}g} e^n$, $(K, L)$ is a <u>finite</u> simplicial pair, and $(|K|, |L|) \xrightarrow{f} (X, A)$ is some map. There exist a subdivision $(K', L')$ of $(K, L)$ and a map $f' : (|K|, |L|) \longrightarrow (X, A)$ s.t.

(i) $f|_{f^{-1}(A)} = f'|_{f^{-1}(A)}|$ and $f \simeq_{\text{rel } f^{-1}(A)} f'$, and

(ii) ~~if~~ for $\sigma \in K'$ if $f'(|\sigma|)$ meets $\mathring{e}^n$ then $f'(|\sigma|) \subseteq \mathring{e}^n$ and $f'|_{|\sigma|}$ is a linear map. $\square$

## The homotopy theory of CW complexes I (6.3-6.28)

An important feature of CW complexes is that their homotopy type depends only on their assembly data up to homotopy.

Lem: Given two maps $S^{n-1} \xrightarrow{\omega_1, \omega_2} X$, a homotopy $H: \omega_1 \sim \omega_2$ begets a homotopy equivalence $X \cup_{\omega_1} CS^{n-1} \xrightarrow{\simeq} X \cup_{\omega_2} CS^{n-1}$.

Pf: Subdivide $CS^{n-1}$ into an outer annulus ($[1/2, 1]$ in the cone coordinate) + an inner disk. Extend the identity map on $X$ by running the homotopy $H$ on the annulus, then using the homeo$^{\underline{m}}$ $S^{n-1} \wedge [0, 1/2] \cong CS^{n-1}$ to cover the $n$-cell in the target. □

In fact, CW complexes are very homotopically well-behaved. Our goal is to get familiar with some of these facts today + to compute $\pi_{* \leq n} S^n$.

↖ not prove!

Lem: For $(X, A)$ a relative CW-complex, $(X, (X, A)^n)$ is $n$-connected. □

Cor: The inclusion $X^n \hookrightarrow X$ is $n$-connected. □

Cor: $\pi_{<n} S^n = 0$.

Pf: There's a cell structure on $S^n$ with $(S^n)^{n-1} = \{s_0\}$, and we have a lexseq of relative homotopy groups that shows
$$\pi_{*<n} S^n = \pi_{*<n} (S^n)^{n-1} = \pi_{*<n} \{s_0\} = 0. \quad □$$

converse ⇝ to prev. Lem.

Lem: If $(X, A)$ is $n$-connected, then $\exists$ an equiv$^{\underline{ce}}$ $(X, A) \sim (X', A')$ with $(X', A')^n = A'$. (Try setting $A = *$.) □

Cor: For $X$ $n$-connected and $Y$ $m$-connected, $X \wedge Y$ is $(n+m+1)$-conn$^{\underline{d}}$.

Pf: The cells in $X \times Y$ take the form $* \times *$, $* \times e^{mj}_\beta$, $e^i_\alpha \times *$, and $e^i_\alpha \times e^j_\beta$. All the former are in $X \vee Y$, so the first non-trivial cell in $X \wedge Y$ lies in dimension $n+m+2$. □

Thm (Homotopy excision): Take $A, B \subseteq X$ with $(A, A\cap B)$ $n$-conn$^{d}$ and $(B, A\cap B)$ $m$-conn$^{d}$. Then $\pi_*(A, A\cap B) \longrightarrow \pi_*(X, B)$ is an iso$^{\underline{m}}$ for $* < n+m$ and an epi$^{\underline{m}}$ for $* = n+m$. □

$$\begin{tikzcd}
(X,*) \arrow[r] \arrow[d] & (X\cup_A CA, CA) \arrow[d, "\cong"] \\
(X/A,*) \arrow[r, "\cong"] & (X\cup_A CA/CA,*)
\end{tikzcd}$$

Cor: If $(X,A)$ is $n$-connected and $A$ is $m$-connected, then $\pi_*(X,A) \longrightarrow \pi_*(X/A)$ is an iso$^{\underline{m}}$ for $1 < * \leq n+m$ + epi at $n+m+1$. □

Cor:

$$\begin{tikzcd}
\pi_{*+1}(CX, X) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \pi_{*+1}(CX/X) \arrow[d, phantom, sloped, "\cong"] \\
\pi_* X \arrow[r] & \pi_{*+1}\Sigma X
\end{tikzcd}$$

for $n$-conn$^{d}$ $X$ is an iso$^{\underline{m}}$ for $* \leq 2n$ and an epi$^{\underline{m}}$ for $* = 2n+1$. □

Ex: There is a fibration

$$\begin{tikzcd}
\mathbb{C}^{\times} \arrow[r, hook] \arrow[d, phantom, sloped, "\simeq"] & \mathbb{C}^{n}\setminus 0 \arrow[r] \arrow[d, phantom, sloped, "\simeq"] & \mathbb{C}P^{n-1} \arrow[d, phantom, sloped, "\simeq"] \\
S^1 \arrow[r] & S^{2n-1} \arrow[r] & \mathbb{C}P^{n-1}.
\end{tikzcd}$$

Since $S^{2n-1}$ is $(2n-2)$-conn$^{d}$, $\pi_{*+1}\mathbb{C}P^{n-1} \cong \pi_* S^1$ for $* \leq (2n-1)$. However, $(\mathbb{C}P^{n-1}, \mathbb{C}P^1)$ is 2-connected, so $\pi_2 \mathbb{C}P^1 \cong \pi_2 \mathbb{C}P^{n-1} \cong \cancel{\mathbb{Z}}\, \pi_1 S^1 \cong \mathbb{Z}$. This feeds into Freudenthal: $\pi_n S^n \cong \pi_{n+1} S^{n+1}$ for $n \geq 2$, so $\pi_n S^n \cong \mathbb{Z}$ $\forall n \geq 1$. □

Note: That seq$^{\underline{ce}}$ is a fibration using Grassmannian:

$$\begin{tikzcd}
U(1) \arrow[r] \arrow[d, phantom, sloped, "\simeq"] & \frac{U(n)}{U(n-1)} \arrow[r] \arrow[d, phantom, sloped, "\simeq"] & \frac{U(n)}{U(n-1)\times U(1)} \arrow[d, phantom, sloped, "\simeq"] \\
S^1 \arrow[r] & S^{2n-1} \arrow[r] & \mathbb{C}P^{n-1}
\end{tikzcd}$$

# The homotopy theory of CW-complexes II (6.29–)

The following technical lemma appeared in our study of the relative homotopy long exact seq$^{\underline{ce}}$ of a pair $(Y, B)$:

(3.14) **Lem**: For all

$$\begin{tikzcd} B \arrow[r, hook] & Y \\ S^{n-1} \arrow[u, "\omega|_{S^{n-1}}"] \arrow[r, hook] & D^n \arrow[u, "\omega"'] \end{tikzcd}$$

there exists filler

$$\begin{tikzcd} B \arrow[r, hook] & Y \\ S^{n-1} \arrow[u] \arrow[r, hook] & D^n \arrow[u] \arrow[ul, "\omega'"] \arrow[ul, phantom, shift right=3, "\Leftarrow"] \end{tikzcd}$$

. □

This can be augmented in two ways:

**Lem**: If $f : Z \to Y$ is an $n$-equiv$^{\underline{ce}}$ and $\dim(X, A) \leq n$, then for all

$$\begin{tikzcd} Z \arrow[r, "f"] & Y \\ A \arrow[u, "g"] \arrow[r, hook] & X \arrow[u, "h"'] \end{tikzcd}$$

there exist fillers

$$\begin{tikzcd} Z \arrow[r] & Y \\ A \arrow[u] \arrow[r, hook] & X \arrow[u] \arrow[ul] \arrow[ul, phantom, shift right=3, "\Leftarrow"] \end{tikzcd}$$

. □

**Cor**: For $f : Z \to Y$ an $n$-equiv$^{\underline{ce}}$, $\forall X$ with $\dim X \leq n$, the map $[X, Z] \to [X, Y]$ is onto. If $\dim X < n$, it's an iso$^{\underline{m}}$. □

**Cor (Whitehead)**: A weak equivalence $f : Z \to Y$ of CW-complexes is a homotopy equivalence. □

Filtering a CW-complex by its skeleta and applying the lemma yields another useful result:

**Cor**: All maps $(X, A) \xrightarrow{f} (Y, B)$ of CW-complexes are homotopic (rel $A$) to cellular maps (i.e., $f(X, A)^n \subseteq (Y, B)^n$), and homotopies between cellular maps admit cellular replacements. □

Now recall an observation from last time: for $X$ $n$-connected and $Y$ $m$-connected, we have $(X \times Y, X \vee Y)^{n+m+1} = (X \vee Y)^{n+m+1}$.

**Cor**: For $* \leq n+m$, $\pi_*(X \vee Y) \xrightarrow{\cong} \pi_*(X \times Y)$. □

**Cor**: For $n \geq 2$, $\pi_n(\bigvee_\alpha S^n_\alpha) \cong \bigoplus_\alpha \pi_n S^n_\alpha$. (Also, $\pi_1 \bigvee_\alpha S^1_\alpha \cong *_\alpha \pi_1 S^1_\alpha$.) □

<u>Lem</u>: For any abelian group $A$, and index $n \geq 2$, there exist a CW-complex $K(A,n)$ with $\pi_* K(A,n) = \begin{cases} A & \text{if } * = n, \\ 0 & \text{o/w}. \end{cases}$

Pf: Select a presentation $\mathbb{Z}^J \overset{f}{\hookrightarrow} \mathbb{Z}^I \twoheadrightarrow A$. We model the middle node as $\bigvee_I S^n$, and the <u>Cor</u> gives a map $\bigvee_J S^n \xrightarrow{\tilde{g}} \bigvee_I S^n$ inducing $g$ on $\pi_n$. The cone on $g$ gives a complex $X_n$ with $\pi_{*<n} X_n = 0$ and $\pi_n X_n = A$. We inductively form $X_{n+j+1}$ from $X_{n+j}$ by killing the homotopy in degree $n+j+1$ by taking more mapping cones. □

<u>Lem</u>: For $X$ with $\pi_{*<n} = 0$ and $Y$ with $\pi_{*>n} = 0$, homotopy classes $[X,Y]$ biject with homomorphisms $\pi_n X \to \pi_n Y$.

Pf: There exist a CW model of $X$ with $X^{n-1} = *$, so we have

$$\begin{tikzcd}
\bigvee_I S^n \arrow[r, Rightarrow] & X^n \arrow[r] & X^{n+1} \arrow[r] & X^{n+2} \arrow[r] & \cdots \arrow[r] & X. \\
& \bigvee_J S^n \arrow[u] & \bigvee S^{n+1} \arrow[u] & \cdots \arrow[u] & &
\end{tikzcd}$$

Maps into $Y$ can be constructed inductively: we begin with $\bigvee_I S^n \to Y$, then we need the precomposite to vanish $\bigvee_J S^n \to \bigvee_I S^n \to Y$ to guarantee an extension, the unicity of which is measured by $[\Sigma \bigvee_J S^n, Y] = 0$. This only gets easier as we go up to skeletal tower. □

<u>Cor</u>: $K(A,n)$ is independent of choice of presentation. □

<u>Rem</u>: $\Omega K(A,n) \cong K(A,n-1)$, and $\Sigma K(A,n-1) \to K(A,n)$. ← interesting map.

<u>Rem</u>: $\pi_n(X^n, X^{n-1})$ is free on generators $\gamma \cdot [f^n_\alpha]$, $\gamma \in \pi_1 X^{n-1}$ and $f^n_\alpha$ a characteristic map of ~~an~~ $n$-cell.

<u>Rem</u>: For <u>all</u> spaces $X$, there exist a ~~space~~ CW-cplx $\tilde{X} \to X$ such that the map is a weak equiv$^{\underline{ce}}$.

## Brown representability (9)

Early on, our description of the pasting lemma was that ~~Ɛ~~ $\mathsf{Spaces}(-,T)$ satisfied a sheaf condition. Today we prove a converse in the homotopy category.

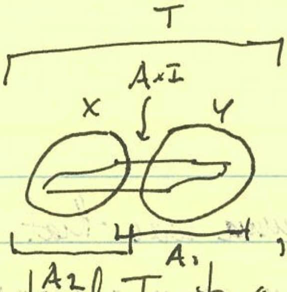
Thm (Brown): Take $F : h\mathsf{Spaces}^{op}_{conn,*} \longrightarrow \mathsf{Sets}_*$, satisfying
(i) a wedge axiom: $F(\vee_\alpha X_\alpha) \cong \prod_\alpha F(X_\alpha)$, and
(ii) a sheaf condition: if $X = A_1 \cup A_2$, $f_1 \in F(A_1)$, $f_2 \in F(A_2)$, and $f_1|_{A_1 \cap A_2} = f_2|_{A_1 \cap A_2}$, then $\exists f \in F(X)$, $f_1 = f|_{A_1}$, $f_2 = f|_{A_2}$.
There then exist a CW-complex $Y$ + an element $u \in F(Y)$ such that $[T, Y] \longrightarrow F(T)$, $\varphi \longmapsto \varphi^*(u)$, is a natural bij$^n$, and there is a compatible ~~translation~~ bijection between natural transformations $F \longrightarrow F'$ and homotopy classes $Y \longrightarrow Y'$.

We construct this in stages. Say an element $u \in Y$ is <u>n-universal</u> if $[S^q, Y] \longrightarrow F(S^q)$ is onto for $q \underline{\leq} n$ and iso for $q < n$. (Note that these are the building blocks of CW-complexes of dim$^n \leq n$.)
Suppose we have an $n$-universal element $u_n$ on a complex $Y$; we set about trying to fix $[S^{n+1}, Y] \longrightarrow F(S^{n+1})$ and $[S^n, Y] \longrightarrow F(S^n)$.

sufficient b/c $S^{q>0}$ is an H-cogroup! $\rightsquigarrow$

Consider $\{\alpha \in \pi_n Y \mid \alpha^* u_n = 0\} = A$ and $L = F(S^{n+1})$, and form $\vee_\alpha S^n_\alpha \xrightarrow{\alpha} Y \vee \vee_\lambda S^{n+1}_\lambda \longrightarrow Y'$, the mapping cone. Applying $F$, we have $(u_n \vee \vee_\lambda \lambda) \longmapsto \vee_\alpha \alpha^*(u_n) = 0$, hence $\exists\, u_{n+1} \in F(Y')$. Since $Y'$ is formed from $Y$ using $(n+1)$-cells, it agrees with $Y$ on $\pi_{<n}$, and it is fixed at $\pi_n$ exactly. The wedge over $\lambda$ gives surj$^{ty}$ at $\pi_{n+1}$.

Lem: For $Y$ a space w/ univ$^l$ element $u$, $(X, A)$ a CW-pair, and $v \in F(X)$, cellular $g : A \to Y$ classifying $v|_A$, then $\exists$ a cellular map classifying $v$ and extending $g$.



<u>Pf idea</u>: , $u \in F(A_1)$ and $v \in F(A_2) \rightsquigarrow w \in F(T)$.
We can extend $T$ to a CW pair $(Y';T)$ with universal element $u'$ restricting to $w$ (and hence to $u$). The induced weak equiv$^{ce}$ $Y \hookrightarrow Y'$ + Whitehead's theorem gives the map $X \hookrightarrow Y' \longrightarrow Y$. □

<u>Pf of Thm</u>: To get surjectivity, set $A = \{x_0\}$ in the above. To get injectivity, set $X' = X \times I$ and $A' = X \times \partial I$. To get the statement about natural transformations, consider

$$\begin{tikzcd}
\mathrm{id} \in {[Y,Y]} \arrow[d, "\cong"] \arrow[d, mapsto, bend right] & {[Y',Y']} \arrow[d, "\cong"] \ni f \\
F(Y) \arrow[r, "T"] \arrow[r, mapsto, bend right] & F'(Y) \arrow[u, mapsto, bend right]
\end{tikzcd}$$
. □

There is a useful companion result that works with functors $F$ defined only on <u>finite</u> CW-complexes. The idea is to define $\hat{F}(X) = \overset{\alpha}{\varprojlim} F(X_\alpha)$, which satisfies the wedge axiom, ~~[illegible]~~ only a weak form of Mayer-Vietoris, but the usual projection $F(\overset{\alpha}{\varinjlim} X_\alpha) \twoheadrightarrow \overset{\alpha}{\varprojlim} F(X_\alpha)$ is an iso$^{m}$.

<u>Thm</u> (Adams): If $F$ is a functor to <u>groups</u> ~~[illegible]~~ from <u>finite</u> CW complexes satisfying (i) and (ii), then it is representable. Natural transformations induce maps of representing objects that are unique ∃ up to <u>weak homotopy</u>: ~~for any~~ restricting along an incoming map from <u>any</u> finite complex gives two homotopic maps. □

<u>Ex</u>: E-M spaces arise from these constructions applied to $H^n(-;A)$.

## Spectra (8-8.32)

We remarked at the end of last time that Brown Rep. applied to $H^n(-;A)$ gives $K(A,n)$, and a sub-claim of that is that $H^n(-;A)$ satisfies the wedge and Mayer-Vietoris axioms. In fact, this is most of what it means to be a homology theory; the one remaining axiom is:

<u>Suspension axiom</u>: There is a natural iso$^{\underline{m}}$ $\underset{\cong}{H^n(X)} \xrightarrow{\cong} \underset{\cong}{H^{n+1}(\Sigma X)}$.

Coupling this to representability gives $[X, K(A,n)] \xrightarrow{\cong} [\Sigma X, K(A,n+1)]$,

an adjunction juggle gives $[X, K(A,n)] \xrightarrow{\cong} [X, \Omega K(A,n+1)]$,

and the Yoneda lemma gives a map $K(A,n) \xrightarrow{\cong} \Omega K(A,n+1)$,

or, equivalently, a map $\Sigma K(A,n) \longrightarrow K(A,n+1)$.

We would like a category of such systems, which has the following:

① Cohomology theories live in this category as single objects.

② Spaces map into this category such that $[\text{"}X\text{"}, \text{"}E_n\text{"}] \cong E^n(X)$.

③ This embedding of spaces is compatible with the connectivity-stabilized theorems from Ch. 6 — for instance, $\pi_* \text{"}X\text{"} \cong \pi_{*+1} \text{"}\Sigma X\text{"}$ $\forall *$.

Def: A <u>spectrum</u> is a collection $\{E_n\}_n$ of CW-complexes such that $\Sigma E_n$ is (homeo$^{\underline{c}}$ to) a subcomplex of $E_{n+1}$ $\forall n$. (Note that the mapping cylinder construction will make the "subcomplex" condition empty.)

Ex: $X$ a space gives $(\Sigma^\infty X)_n = \Sigma^n X$.

Ex: the <u>Eilenberg-MacLane spectrum</u> $(HA)_n = K(A,n)$.

Maps between spectra are harder to define. For instance, we know $\pi_n S^n \cong \mathbb{Z}$ for $n \geq 1$, but $\pi_0 S^0 = \{\pm 1\}$, and so if we were to define maps of spectra as commuting seq$^{\underline{ces}}$

$$\begin{tikzcd} \Sigma S^n \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \Sigma S^n \arrow[d, phantom, "\cong" rotate=90] \\ S^{n+1} \arrow[r] & S^{n+1} \end{tikzcd},$$

we would get $\pi_0 \Sigma^\infty \mathbb{S}^0 = \{\pm 1\}$ — the <u>wrong</u> answer.

The solution is to ask for maps to only be defined _eventually_.

Def: A _subspectrum_ $F \subseteq E$ is a seq$^{\underline{ce}}$ of subcomplexes of $E_n$, forming a spectrum by restriction. It is _cofinal_ when every cell $e^m_\alpha \subseteq E_n$ has $\Sigma^{j_\alpha} e^m_\alpha \subseteq F_{n+j_\alpha}$ — it eventually appears in $F$. A map $E \longrightarrow E'$ is required only to be defined on a cofinal $F \subseteq E$, and two maps are equal if they agree on a mutually cofinal subspectrum.

Rem: The inclusion of a cofinal subspectrum is equivalent to the identity map.

Def: Two maps of spectra are _homotopic_ if there is a common cofinal subspectrum $F'$ and a map $F' \wedge I_+ \longrightarrow E'$ witnessing the homotopy.

Def: Spectra have _wedge sums_, given level-wise, and _mapping cones_, as given ~~as~~ level-wise.

This is enough to copy the proof (which we did not give) of Whitehead:

Thm: Set $\pi_n E = [\Sigma^\infty S^n, E] \cong \varinjlim \pi_{n+k} E_k$. If a map $f : E \to E'$ induces a weak equiv$^{\underline{ce}}$, then it's a homotopy equiv$^{\underline{ce}}$. □

Cor: The spectra $\{E_n \wedge S^1\}_n$ and $\{E_{n+1}\}_n$ are equivalent. □

Cor: The spectrum $\{E_{n-1} \wedge S^1\}_n$ is equivalent to $E$. □

Cor: $[E, E']$ is an abelian group, since $[E, E'] \cong [\Sigma^2 E, \Sigma^2 E']$. □

Spectra are also set up to short circuit Brown representability:

Lem: $\mathrm{Spectra}(\Sigma^\infty X, E) \cong \mathrm{Spaces}(X, \Omega^\infty E)$ where $\Omega^\infty E = \varinjlim (\Omega^k E_k)$.

Pf: Definitionally, 
$$\begin{aligned}\mathrm{Spectra}(\Sigma^\infty X, E) &= \varinjlim^n \mathrm{Spaces}(\Sigma^n X, E_n) \\ &= \varinjlim^n \mathrm{Spaces}(X, \Omega^n E_n) \\ &= \mathrm{Spaces}(X, \varinjlim^n \Omega^n E_n)\end{aligned}$$
when $X$ is CW. □

## Co/homology theories from spectra (8.33-)

We defined spectra in such a way that a cohomology theory gives rise to a spectrum. Our definition was lax enough, though, that the converse is not quite as clear.

**Def:** For a spectrum $E$, we define two functors $\mathrm{Spaces}_*^{(op)} \longrightarrow \mathrm{AbGroups}$:
$\tilde{E}_n(X) = \pi_n(E \wedge X)$ and $\tilde{E}^n(X) = [\Sigma^\infty X, \Sigma^n E] = [\Sigma^{-n}\Sigma^\infty X, E]$,
where $(E \wedge X)_n = E_n \wedge X$ is induced up from spaces.

In order to see that there are co/homology functors, it's useful to record:

**Lem:** $[\Sigma^\infty X, \Sigma^\infty Y] = \varinjlim_{m\to\infty} [\Sigma^m X, \Sigma^m Y]$, and $E = \varinjlim_{n\to\infty} \Sigma^{-n}\Sigma^\infty E_n$,
so $[\Sigma^\infty X, E] = \varinjlim_{n,m\to\infty} [\Sigma^m X, \Sigma^{m-n} E_n]$. □

**Rem:** In general, homotopy classes of maps of spectra are presented by $\pi_0$ of a kind of pro-ind-space.

Now we check the axioms:

① We've built in suspension-invariance:
$\tilde{E}_{n+1}(\Sigma X) \cong [S^{n+1}, E \wedge \Sigma X] \cong [S^n, E \wedge X] = \tilde{E}_n(X)$, and
$\tilde{E}^{n+1}(\Sigma X) = [\Sigma^\infty \Sigma X, \Sigma^{n+1} E] \cong [\Sigma^\infty X, \Sigma^n E] = \tilde{E}^n(X)$.

② Cofiber sequences $A \xrightarrow{i} X \to X \cup_i CA$ are converted to lexseq$^{s}$:
$E \wedge A \longrightarrow E \wedge X \longrightarrow E \wedge (X \cup_i CA) = (E \wedge X) \cup_i C(E \wedge A)$ is again coexact,
so $\pi_n E \wedge A \longrightarrow \pi_n E \wedge X \longrightarrow \pi_n E \wedge C(i)$ is exact. For cohomology,
$\Sigma^\infty A \longrightarrow \Sigma^\infty X \longrightarrow \Sigma^\infty(X \cup_i CA)$ is coexact, so mapping into $E$ is exact.

③ The cohomological wedge axiom is easy: $[\bigvee_\alpha \Sigma^\infty X_\alpha, E] = \prod_\alpha [\Sigma^\infty X_\alpha, E]$ because coproducts pull out on the left to products. Homology is harder and requires a filtration trick: homology satisfies the finite wedge axiom by ② and smash products commute with colimits, so
$$E \wedge \varinjlim_{\substack{S \subseteq A \\ \text{finite}}} \bigvee_{\alpha \in S} X_\alpha \cong \varinjlim_S E \wedge \bigvee_{\alpha \in S} X_\alpha \cong \varinjlim_S \bigvee_{\alpha \in S} E \wedge X_\alpha \cong \bigvee_\alpha E \wedge X_\alpha. \qquad \square$$

Rem: The suspension invariance of the individual spaces $E_n$ extracted from Brown rep. of a cohomology functor makes the system in the Lemma constant. "Λ-spectrum"

Rem: Cohomology does not commute with colimits. Instead, there is a Milnor seq^ce:

$$0 \longrightarrow \varprojlim_{\alpha}{}^{1} E^{n-1}(X_\alpha) \longrightarrow E^n(\varprojlim_{\alpha} X_\alpha) \longrightarrow \varprojlim_{\alpha} E^n(X_\alpha) \longrightarrow 0.$$

Rem: To check that these two constructions CohThys $\rightleftarrows$ Spectra are inverses requires Whitehead's theorem in Spectra + what amounts to a spectral seq^ce argument in CohThys. We'll delay this for a moment.

We can lift maps of cohomology theories up to maps of spectra, granting the remark. A map $E^* \xrightarrow{f} F^*$ induces by Brown rep. a unique sequence of maps $E_n \xrightarrow{f_n} F_n$ and hence a map $\tilde{f}: E \longrightarrow F$ of spectra. It's even natural to extend the definition of cohomology to $F^0(E) = [E, F]$, of which $\tilde{f}$ is a natural element.

Ex: The spectrum $HA$ represents cohomology w/ coeff^s in $A$.

Ex: The spectrum $\mathbb{S} := \Sigma^\infty$ has associated homology theory stable homotopy.

Ex: The functor $X \longmapsto (\pi_*^s X) \otimes \mathbb{Z}_{(p)}$ is exact, hence has an associated spectrum $\mathbb{S}_{(p)}$, the $p$-local sphere spectrum.
(In fact, this trick works for any homology theory.)

Ex: The functor $X \longmapsto AbGps(\pi^s_{*0} X, \mathbb{Q}/\mathbb{Z})$ is exact, hence has an associated spectrum $\mathbb{I}$ with $[X, \mathbb{I}] = 5$. This is called the Brown-Comenetz dualizing object.

We'll probably find some more as we go.

Thm (Hurewicz): There is a map $\mathbb{S} \longrightarrow H\mathbb{Z}$ with 0-connected fibers
$\Longrightarrow$ the difference between $\mathbb{S}_*(X)$ and $H\mathbb{Z}_*(X)$ begins one degree above the bottommost nonzero group in $H\mathbb{Z}_*(X)$ $\Longrightarrow$ for $X$ n≥1-connected,
$\pi_n X \cong \pi_n^s X \cong H\mathbb{Z}_n X$. ☐

Spectral Sequences (Boardman's Conditionally Convergent...)

These arise everywhere in algebraic topology, and they are as fundamental as the notion of a long exact sequence. Lexseq$^s$ appear when calculating $H_*$ of $A \overset{j}{\hookrightarrow} X \longrightarrow X \cup_j CA$. What if there are many coexact seq$^s$ constructing $X$?

$$\begin{tikzcd}
\ast \arrow[r] & H_*X_1 \arrow[r] & H_*X_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] \arrow[dl, dotted] & \overbrace{H_*X_n}^{D^1_{*,n}} \arrow[r] \arrow[dl, dotted] & H_*X_{n+1} \arrow[r] \arrow[dl] & \cdots \arrow[r] \arrow[dl, dotted] & H_*X \\
 & H_*A_1 \arrow[u] & H_*A_2 \arrow[u] \arrow[l] & \cdots \arrow[l, dotted] & \underbrace{H_*A_n}_{E^1_{*,n}} \arrow[u] \arrow[l, dotted] & H_*A_{n+1} \arrow[u] \arrow[l] & \cdots \arrow[l] &
\end{tikzcd}$$

Suppose we want to recover $H_*X$ from this setup. Start by noticing $H_*X = \varinjlim^n (H_*X_n)$, so that $x \in H_*X$ arises as $\tilde{x} \in H_*X_n$ for some $n$. Note also that at the minimal such $n$, it pushes down to give a nonzero class $a_{n-1} \in H_*A_{n-1}$.

What about the other classes $a_{n-1} \in H_*A_{n-1}$? When do they come from $x_n$?

$$\begin{tikzcd}
 & & x_n \arrow[dddl, bend right] & (\exists x_{n+1}??) \arrow[dddl] & \\
\cdots \arrow[r] & H_*X_{n-1} \arrow[r] & H_*X_n \arrow[r] \arrow[dl] & H_*X_{n+1} \arrow[r] \arrow[dl] & \cdots \\
 & H_*A_{n-1} \arrow[u] & H_*A_n \arrow[u] \arrow[l] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & H_*A_{n+1} \arrow[u] \arrow[l] & \\
 & d_1(a_n). & a_n \arrow[uuu, mapsto, bend left] & &
\end{tikzcd}$$

Note that if $d_1(a_n) \neq 0$, then $x_n \neq 0$, hence $a_n \notin \ker(H_*A_n \to H_*X_n)$, and $\nexists x_{n+1}$.
On the other hand...

$$\begin{tikzcd}
 & & \exists x_{n-1} \arrow[r, mapsto] \arrow[dddll] & x_n \arrow[dddl, bend right] & & \\
\cdots \arrow[r] & H_*X_{n-2} \arrow[r] & H_*X_{n-1} \arrow[r] \arrow[dl] & H_*X_n \arrow[r] \arrow[dl] & H_*X_{n+1} \arrow[r] \arrow[dl] & \cdots \\
 & H_*A_{n-2} \arrow[u] & H_*A_{n-1} \arrow[u] \arrow[l] & H_*A_n \arrow[u] \arrow[l] & H_*A_{n+1} \arrow[u] \arrow[l] & \\
d_2(a_n) & & 0 & a_n \arrow[l, mapsto] \arrow[uuu, mapsto, bend left] & &
\end{tikzcd}$$

I. Claim: This assignment is well-defined up to im $d_1$, and hence it determines a function $d_2 : H_*(H_*A_*; d_1) \longrightarrow H_*(H_*A_{*-2}; d_1)$.

II. Claim: This process continues ad infinitum. Since $X_0 = *$, eventually $x_0 = 0$ is guaranteed, but then $x_j = 0 \ \forall j \leq n$, hence $\exists x_{n+1}$. ☺

III. Claim: The surviving elements in a spectral seq$^{\underline{\text{ce}}}$ are the associated graded of a filtration of $H_*X$ (by minimal lift degree).

Rem: This story is complicated some by ① using a bi-infinite filtration, or ② using a cohomology functor / a descending filtration, primarily because such spectral seq$^{\underline{\text{ces}}}$ need not stabilize. You have to incorporate taking inverse limits of the subquotients of $H_*A_*$, which can destroy some of the exactness in III, or the argument used in II.

Lem: If a map of spectral seq$^{\underline{\text{ces}}}$ is ever an iso$^{\underline{\text{m}}}$ on $H_*(H_*A_*, d_n)$, it is an iso$^{\underline{\text{m}}}$ forever after.
$\Longrightarrow$ their targets are iso$^{\underline{\text{ic}}}$ by the same map. □

Ex: Filter $X$ by skeleta, so $A_n = \vee_\alpha S^n_\alpha$ and $H_*A_n = (\Sigma^n G)^{\oplus \alpha}$.
The map $d_1$ is

$$\begin{tikzcd}
H_*X_{n-1} \arrow[r] & H_*X_n \arrow[dl] \\
H_* \cancel{A_{n-1}} \vee_\beta S^{n-1}_\beta \arrow[u] & H_* \cancel{A_n} \vee_\alpha S^n_\alpha \arrow[u] \arrow[l, "d_1"']
\end{tikzcd}$$

exactly the cellular differential,

so that $H_*(H_*A_*; d_1) = H^{cell}_*(X)$. All higher diff$^{\underline{\text{ls}}}$ are zero because $\bigoplus_\alpha \Sigma^n G \xrightarrow{[-1]} \bigoplus_\gamma \Sigma^{n-r} G$ has the wrong degree. We say the SS collapses at $E_2$, and this is a proof the cellular homology computes homology.

Cor: More generally, if $E + F$ satisfy Eilenberg-Steenrod and $E_*(S^n) \longrightarrow F_*(S^n)$ is an iso$^{\underline{\text{m}}}$ $\forall n$, then $E_*(X) \longrightarrow F_*(X)$ is an iso$^{\underline{\text{m}}}$ for all CW complexes $X$. □

## Obstruction Theory

Obstruction theory is generally concerned with trying to extend a diagram like
$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, dashed] & B \\ X \arrow[r, dashed] \arrow[ur, dashed] & Y \arrow[u] \end{tikzcd}$$
in any of the indicated dashed ways.

We are going to summarize our results in terms of computing $\pi_0 Y^X$, leaving the relative cases of $A \hookrightarrow X$ and $Y \twoheadrightarrow B$ to the reader.

Recall one of our older Lemmas:

Lem: If $\pi_{*<n} Y = 0$ and $\pi_{*>n} Z = 0$, then $[Y, Z] \xrightarrow{\simeq} [\pi_n Y, \pi_n Z]$. □

Cor: If $Y$ is $(n-1)$-connected, there is a canonical map $Y \to K(\pi_n Y, n)$ inducing an iso$^{\underline{m}}$ on $\pi_n$. □

Cor: For $Y$ $(n-1)$-connected, the fiber of $Y \longrightarrow K(\pi_n Y, n)$ witnesses the $(n+1)$-truncation of $Y$, $Y(n, \infty)$. It has the properties $\pi_* Y(n,\infty) = \{\pi_* Y \text{ if } * > n, 0 \text{ o/w}\}$ and $\pi_* Y(n, \infty) \longrightarrow \pi_* Y$ is an iso$^{\underline{m}}$ for $* \geq n$. □

Successive applications of this lead to the Postnikov tower:
$$\begin{tikzcd} Y \arrow[d] & Y(1,\infty) \arrow[l, "fib"'] \arrow[d] & Y(2,\infty) \arrow[l, "fib"'] \arrow[d] & \cdots \arrow[l] & Y(n,\infty) \arrow[l] \arrow[d] & \cdots \arrow[l] & * \arrow[l] \\ K(\pi_1 Y, 1) & K(\pi_2 Y, 2) & K(\pi_3 Y, 3) & & K(\pi_{n+1} Y, n+1) & & \end{tikzcd};$$
a diagram of interlocking fiber seq$^{\underline{ces}}$. This situation is ripe for a sseq.!

Rem: Applying $\pi_*$ to this diagram gives an extremely boring sseq.

Instead, we apply $\pi_* F(X, -)$, where $X$ is some fixed test space. The functor $F(X, -)$ preserves fiber seq$^{\underline{ces}}$ + $\pi_*$ turns them into lexseq$^{\underline{s}}$, giving

Def: Federer's spectral seq$^{\underline{ce}}$ has signature
$$E^1_{m,n} = \underbrace{\pi_m F(X, K(\pi_{n+1} Y, n+1))}_{H^{n+1-m}(X; \pi_{n+1} Y)} \Longrightarrow \pi_m F(X, Y).$$

~~Rem: The $d_1$ differential is induced by the going-around map~~
~~$\underbrace{\Omega K(\pi_n Y, n)}_{K(\pi_n Y, n-1)} \longrightarrow K(\pi_{n+1} Y, n+1)$, called the $n^{th}$ k-invariant of $Y$.~~
~~Brown identifies these with cohomology operations.~~

Ex: Another interesting case is when $Y = \underline{E}_0$ is an $\infty$-loopspace. This spectral seq$^{\underline{ce}}$ then recovers the AHSS from the previous lecture.

Rem: For arbitrary $X$ and $Y$, $\pi_0$ and $\pi_1$ of $F(X, Y)$ are sets and groups respectively. This situation is called a fringed spectral seq$^{\underline{ce}}$, and they are considerably more obnoxious. Set $X = \Sigma^2 X'$ or $Y = \Omega^2 Y'$ to avoid this situation.

Rem: If $\pi_{*<n} X = 0$ + $\pi_{*>n} Y = 0$, we have $H^{<n}(X; \text{any}) = 0$ and $H^{\text{any}}(X; \pi_{>n} Y) = 0$. This puts a single nonvanishing gp in the 0-line of the sseq:
$H^n(X, \pi_n Y) \cong \text{Hom}(H_n X, \pi_n Y) \cong \text{Hom}(\pi_n X, \pi_n Y)$.

Rem: The relative version of this spectral seq$^{\underline{ce}}$ also recover the lifting lemma about

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & B \\ X \arrow[r, dashed] & Y \arrow[u] \end{tikzcd}$$

for $(X, A)$ $n$-connected, and $(Y, B)$ an $n$-equiv$^{\underline{ce}}$.

What information would we need to understand nontrivial examples of this sseq$^{\underline{ce}}$? The $d_1$-differential is induced by pushforward along the map $k_n$ in

$$\begin{tikzcd} Y[n, \infty) & & Y(n, \infty) \arrow[ll] \arrow[d] \\ K(\pi_n Y, n) \arrow[from=u] & \Omega K(\pi_n Y, n) \arrow[ur] \arrow[r, "k_n"] & K(\pi_{n+1} Y, n+1) \end{tikzcd}$$

This map is part of the homotopy data of $Y$, called the $n^{\underline{th}}$ k-invariant of $Y$. Note $K(\pi_n Y, n-1) \simeq K(\pi_n Y, n) \longrightarrow K(\pi_{n+1} Y, n+1)$ is a map of E-M spaces, so can be considered as the data of a natural transformation $\sigma : H^{n-1}(-; \pi_n Y) \longrightarrow H^{n+1}(-; \pi_{n+1} Y)$.

① What are these? How many are there? How can they be discerned for some random space $Y$?

② This is kind of the "dual problem" to computing $[S^m, S^n]$.

A complicated example:

We haven't yet had an example of a spectral seq$^{ce}$ in which we can compute. This is because it's impossible to find a spectral seq$^{ce}$ that is
① easy ② tangible ③ nontrivial ④ well-motivated,
ll at once. Today we will do ①-③.

For whatever reason, suppose that we are interested in computing power series of the form $1 +$ (higher terms) that satisfy a kind of cocycle condition:

① : $f(x) = 1 + \cdots$ satisfying $f(x+y) / f(x) \cdot f(y) = 1$.

② $g(x,y) = 1 + \cdots$ satisfying $\dfrac{g(x,y) \cdot g(t, x+y)}{g(t+x, y) \cdot g(t,x)} = 1$.

③ ...

Note that these form a cochain complex. We want its cohomology.

Note that this is instantly hard. Try $f(x) = 1 + x$, so $\frac{1+x+y}{(1+x)(1+y)} =$

$(1+x+y)(1+x+y+xy)^{-1} = 1 \cancel{+x} \cancel{+y} \cancel{-x} \cancel{-y} - xy \cancel{-x^2} - xy \cancel{-y^2} - xy + \cancel{x^2} \cancel{+y^2} + \cdots$

$= -3xy + \cdots$, so that didn't work + we have to try again.

Rem: For sanity's sake, let's work over $\mathbb{F}_2$.

Rem: The computation gets much easier "modulo higher terms", since then the denominator is $(1+x)(1+y) = 1 + x + y +$ h.o.t. This isn't computing exactly what we want; it's looking for polynomials of homogeneous degree satisfying additive analogues like

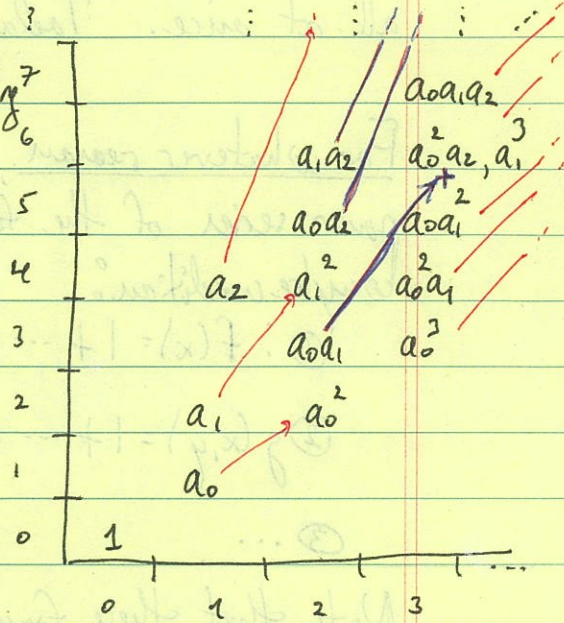
① $f(x) \rightsquigarrow f(x) - f(x+y) + f(y) = 0$.

② $g(x,y) \rightsquigarrow g(x,y) - g(t+x,y) + g(t,x+y) - g(t,x) = 0$.

In degree ①, all the $x^{2^i}$ work. In degree ②, lots of things work: $x^{2^i} y^{2^j}$, for instance, but also $\delta x^7 = x^6y + x^5y^2 + x^4y^3 + x^3y^4 + x^2y^5 + xy^6$.

– cohomology of this complex is $\mathbb{F}_2[a_i \mid i \geq 0]$, $a_i$ rep$^{d}$ by $x^{2^i}$,
homological degree $1$ + poly$^{l}$ degree $2^i$.

can we reconstruct the original question from this?
ctral seq$^{\underline{ce}}$! Filter the original complex by poly$^{l}$ degree,
seq. starting with $\{\text{add. coh.}\}_{*,*} \Longrightarrow \{\text{mult. coh.}\}_*$.



mputed by picking an additive class, building
tive *chain* out of it, and looking at the
e chain diff$^{l}$ in the associated graded.
$= \left[\frac{1+x+y}{(1+x)(1+y)}\right] = [1 + xy + \text{hot}] = a_0a_0.$
, $d_{2^i} a_i = a_i^2$.

$= (1+xy^2)(1+(t+x)y^2)^{-1}(1+t(x+y)^2)(1+tx^2)^{-1}$
$xy^4 + t^2x^2y^2 + \text{hot}.$ ⇝ $d_3(a_0a_1) = a_0^2a_2 + a_1^3$.
$d(a_ia_j) = a_i^2a_{j+1} + a_{i+1}a_j^2$.

diff$^{\underline{ls}}$: $d_3(a_0^3) = a_0^2a_1^2$.
$d_4(a_0^2a_1) = a_1^4$.
$d_5(a_0a_1^2) = a_0^2a_2^2$.
$d_6(a_1^3) = a_1^2a_2^2$
$d_7(a_0a_1a_2) = a_0^2a_2a_3 + a_1^3a_3 + a_1a_2^3$.
$\vdots$

## Smash Products

For all our discussion of homotopy + homology groups, we have not found a framework for the cohomology ring of a space. Since we have constructed an object $HR$ embodying the ordinary cohomology functor, it is reasonable to expect the ring st$^c$ on $R$ to induce structure on $HR$ (which in turn induces it on $H^*(-;R)$). The following observation is key.

Def: A ring is a (comm., unital) monoid in AbelianGroups, using the $\otimes$-product in place of the Cartesian / categorical $\times$.

We, too, would like a monoidal structure on Spectra compatible with our other monoidal structures: $\text{AbGps} \xrightarrow{H} \text{Spectra} \xleftarrow{\Sigma^\infty} \text{Spaces}_{*/}$.

Step 1: For $X$ a space, we have previously defined
$E \wedge X = (E_n \wedge X)_n$ by smashing through levelwise.
This definition is set up so that $\Sigma^\infty X_1 \wedge \Sigma^\infty X_2 = \Sigma^\infty(X_1 \wedge X_2)$.

Step 2: In general, a spectrum is a (sequential, ind-) system of formal desuspensions of suspension spectra. The system "$(E_n)$" might be more honestly written as $\{\cdots \to \Sigma^{-n}\Sigma^\infty E_n \to \Sigma^{-(n+1)}\Sigma^\infty E_{n+1} \to \cdots\}$.
Category theory indicates a useful notion of the $\otimes$ of two such:

$$\begin{tikzcd}
\Sigma^\infty(E_0 \wedge F_0) \arrow[r] \arrow[d] & \Sigma^{-1}\Sigma^\infty(E_0 \wedge F_1) \arrow[r] \arrow[d] & \cdots \\
\Sigma^{-1}\Sigma^\infty(E_1 \wedge F_0) \arrow[r] \arrow[d, dotted] & \Sigma^{-2}\Sigma^\infty(E_1 \wedge F_1) \arrow[r] \arrow[d, dotted] & \cdots \\
{} & {} &
\end{tikzcd}$$

This system is grid-indexed rather than sequential, which we "must" correct.

Bad idea: Identifying ind-systems with their colimits, we might restrict to any cofinal subsystem. However, this destroys e.g. associativity.

Good idea / Step 3: "Sum over possible choices": we set $(E \wedge F)_n$ to be the colimit of the diagram under the $n^{\text{th}}$ antidiagonal (after replacing the maps by cofibrations).

~~Warning~~ Thm: In the homotopy category, this determines a symmetric monoidal product, $\wedge$, on Spectra. □

Warning: This product is not especially nice before passing to the homotopy category. It turns out that this is unavoidable.

Cor: $K(R,n) \times K(R,m) \xrightarrow{\smile} K(R,n+m)$ induces $HR \wedge HR \to HR$. □

Ex: The sphere spectrum, $\mathbb{S}$, is the unit + hence also a ring. □

Rem: In fact, any ring-valued theory acquires a product as in the Cor.

Given a multiplication, we can define the cohomology product like so:

$$\begin{matrix} X \xrightarrow{\omega_n} E_n \\ X \xrightarrow{\omega_m} E_m \end{matrix} \rightsquigarrow \begin{matrix} \Sigma^{-n}\Sigma^\infty X \xrightarrow{\omega_n} E \\ \Sigma^{-m}\Sigma^\infty X \xrightarrow{\omega_m} E \end{matrix} \rightsquigarrow \Sigma^{-(n+m)}\Sigma^\infty X \xrightarrow{\Delta} \Sigma^{-(n+m)}(X \wedge X) \longrightarrow E \wedge E \xrightarrow{\mu} E.$$

Rem: This same product makes $E^*X$ (and $E_*X$) into $E_*$-module(s).

The product structure is also where the duality pairing for co/homology comes from. Given $(\sigma : \mathbb{S}^n \to E \wedge \Sigma^\infty X) \in E_n X$ and $(\omega : \mathbb{S}^m \wedge \Sigma^\infty X \to E) \in E^m X$, we get

$$\langle \omega, \sigma \rangle : \mathbb{S}^{n+m} \xrightarrow{\Sigma^m \sigma} E \wedge \mathbb{S}^m \wedge \Sigma^\infty X \xrightarrow{1 \wedge \omega} E \wedge E \xrightarrow{\mu} E.$$

Under this pairing, the maps $f^*, f_*$ induced by $f : X \to Y$ are adjoint:

$$\begin{tikzcd} \mathbb{S}^{n+m} \arrow[r, "\Sigma^m \sigma"] \arrow[dr, "\Sigma^m f_* \sigma"'] & E \wedge \mathbb{S}^m \wedge \Sigma^\infty X \arrow[d, "1 \wedge 1 \wedge f"] \arrow[dr, "1 \wedge f^* \omega"] & & \\ & E \wedge \mathbb{S}^m \wedge \Sigma^\infty Y \arrow[r, "\omega"] & E \wedge E \arrow[r, "\mu"] & E \end{tikzcd}$$

$$\rightsquigarrow \langle f^*\omega, \sigma \rangle = \langle \omega, f_* \sigma \rangle.$$

Rem: $- \wedge E$ preserves colimits, + there is a version of the adjoint functor theorem that guarantees function spectra. Or, conversely, use Brown rep on $F(E_1, E_2)^0(X) = “\pi_0 F(E_1 \wedge X, E_2)” = \text{Spectra}(E_1 \wedge X, E_2)$.

# he Serre Spectral Seq$^{ce}$

n this section, we introduce one of the most routinely useful pectral seq$^{ces}$ ever encountered, used to analyze the homology of fibration. Its utility comes from how you're not "supposed" to be ble to do this — homology is for cofibrations + homotopy is for fibrations.

ix $F \to E \xrightarrow{f} B$ a fibration over a CW base $B$. The ssociated-graded of $B$ is a wedge of spheres — its cells. Pull back he cellular filtration on $B$ to get a filtration on $E$: $E_n = p^{-1}(B^{(n)})$. he associated-graded of $E$ looks like $E_n / E_{n-1} = (B^{(n)}/B^{(n-1)}) \times F$. ~~and~~ so Applying a homology theory $h_*$ gives a spectral seq$^{ce}$ with $E^1_{*,*} = h_*(E_n, E_{n-1}) = C^{cell}_*(B) \otimes h_*(F)$.

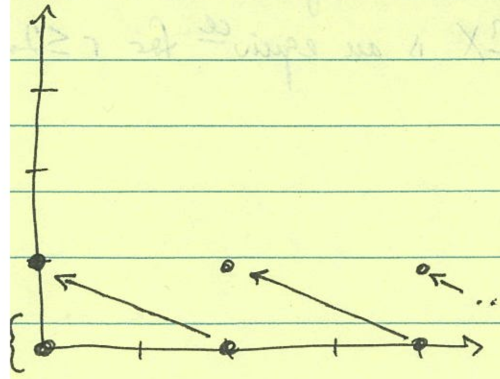
**hm (Serre):** If $\pi_1 B \curvearrowright h_* F$ is trivial (e.g., if $\pi_1 B = 0$), then $d^1_* = d^{cell}$, so $E^2_{p,q} = H_p(B; h_q F) \Longrightarrow h_{p+q}(E)$. $\square$

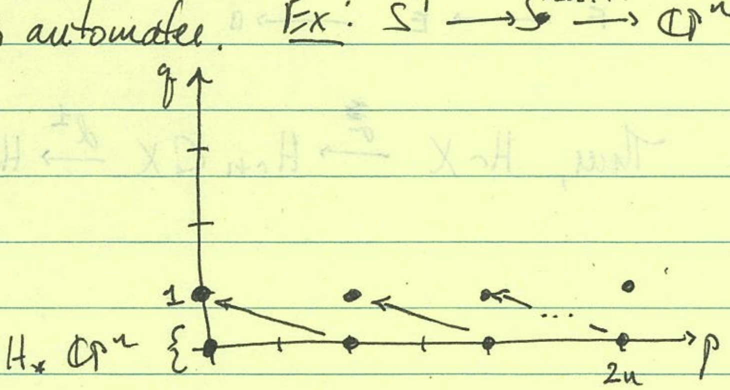
em: There is a version of this without the hypothesis on $\pi_1$, where "local coefficients" / "twisted homology" are used. We won't need it.

um: If $h_*$ is a field, then $H_{p*}(B; h_{q*}F) \equiv H_{p*}(B; h_*) \otimes h_* F$.

t is remarkable how much this automates. Ex: $S^1 \to S^{2n+1} \to \mathbb{CP}^n$.

x: $S^1 \to * \to \mathbb{CP}^\infty$





r construction is <u>natural</u> against maps of fiber seq$^{ces}$ cellular base maps), inducing maps of spectral seq$^{ces}$.

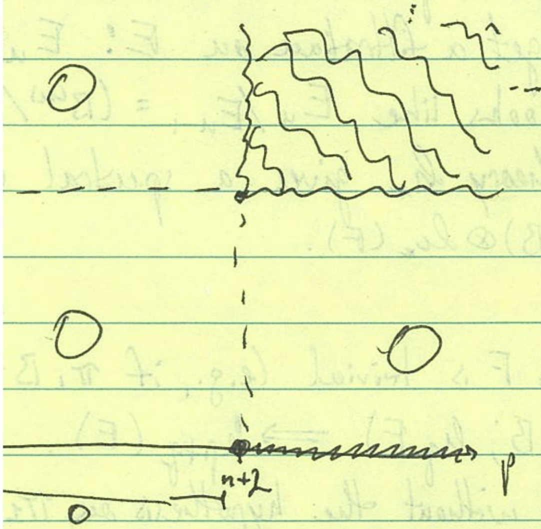
t edge homomorphisms by

$$\begin{tikzcd}
F \arrow[r, equal] \arrow[d] & F \arrow[r] \arrow[d] & * \arrow[d] \\
F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d, equal] \\
* \arrow[r] & B \arrow[r] & B
\end{tikzcd}$$

converging to the behavior of $h_* F \to h_* E \to h_* B$.

udenthal): For $X$ $n$-connected, $X \to \Omega\Sigma X$ is an $n$-equiv$^{ce}$.

iders the SSS for $\Omega\Sigma X \to P\Sigma X \to \Sigma X$:



The only way for this to converge to $H_* P\Sigma X = H_*(pt)$ is for
$d_r : H_r \Sigma X \xrightarrow{\cong} H_{r-1}\Omega\Sigma X$
to be an iso$^{m}$ for $r \leq 2n+2$.

We grab the part of $H_* \Omega\Sigma X$ belonging to $X$ by mapping in a (non-fiber!!) seq$^{ce}$:

$$\begin{tikzcd}
{} \arrow[r] & CX \arrow[r, hook] \arrow[d, "J"] & \Sigma X \arrow[d, equal] \\
{} \arrow[r] & P\Sigma X \arrow[r] \arrow[d, equal] & \Sigma X \arrow[d, equal] \\
{} \arrow[r] & E \arrow[r] & B
\end{tikzcd}$$

inducing

$$\begin{tikzcd}
\tilde{H}_r X \arrow[d, "j_*"] & H_{r+1}(CX,X) \arrow[l, "\cong"', "\partial"] \arrow[r, "\cong"] \arrow[d, "J_*"] & \tilde{H}_{r+1}(\Sigma X) \arrow[d, equal] \\
\tilde{H}_r F & H_{r+1}(E,F) \arrow[l, "\partial"] \arrow[r, "\pi_*"] & H_{r+1}(B) \arrow[ll, bend left, "\text{transgressive } d^{r}"]
\end{tikzcd}$$

$H_r X \xrightarrow{\sigma} H_{r+1} \Sigma X \xrightarrow{d^1} H_r \Omega\Sigma X$ is an equiv$^{ce}$ for $r \leq 2n+2$. □

## Some cohomological computations

The fun of spectral seq$^{\text{ces}}$ in topology is that we don't just filter our problems away into differentials — we then try to compute the differentials. :)

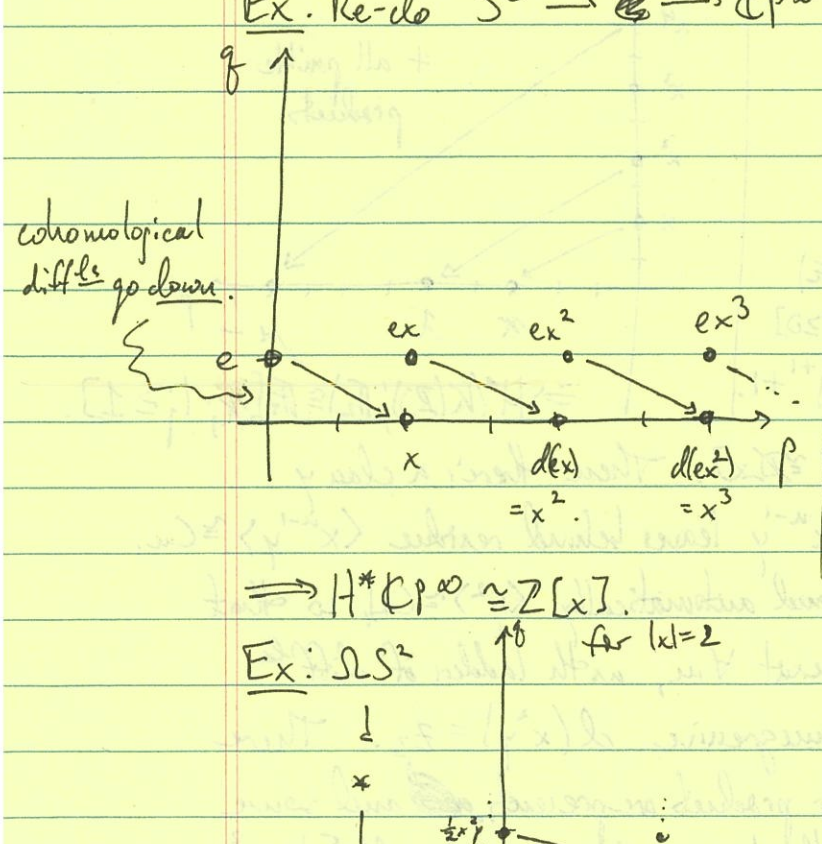
One of our main tools for making computations is the following observation:

**Lem:** The filtration used to form the Serre SS is multiplicative, in the sense that for $F \to E \to B$ and $F' \to E' \to B'$ fiber seq$^{\text{ces}}$, we have $E_n \times E'_m \subseteq (E \times E')_{n+m}$, inducing a pairing $E_r(E) \otimes E_r(E') \to E_r(E \otimes E')$. □

**Cor:** The cohomological Serre SS has a multiplication, restricting to the cup product on the edges, converging to cup on $H^*E$, and satisfying a Leibniz law for the differentials: $d_r(x \cdot y) = (d_r x) \cdot y + (-1)^{|x|} x \cdot d_r(y)$. □

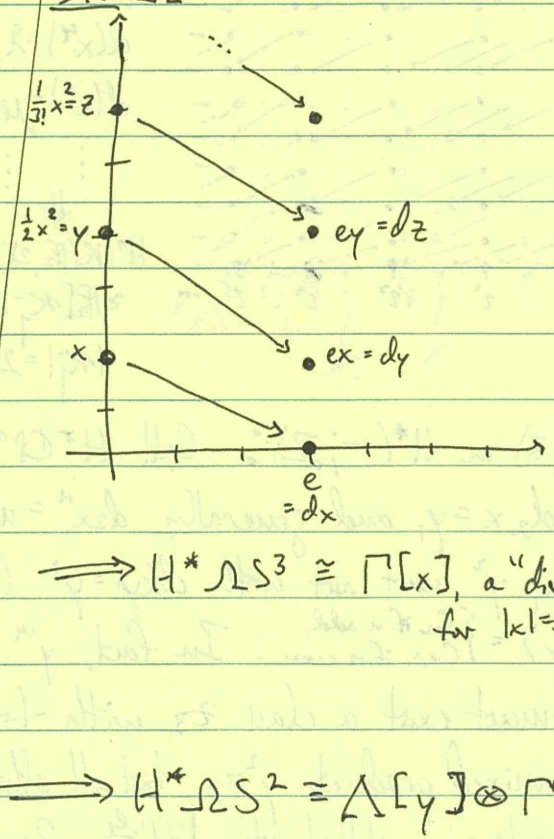
(Everything about this is kind of automatic, except the Leibniz law, which comes out of thinking about the cell structure on $(D^n, \partial D^n) \times (D^m, \partial D^m)$.)

**Ex:** Re-do $S^1 \to * \to \mathbb{C}P^\infty$:



$\Longrightarrow H^*\mathbb{C}P^\infty \cong \mathbb{Z}[x]$. for $|x| = 2$

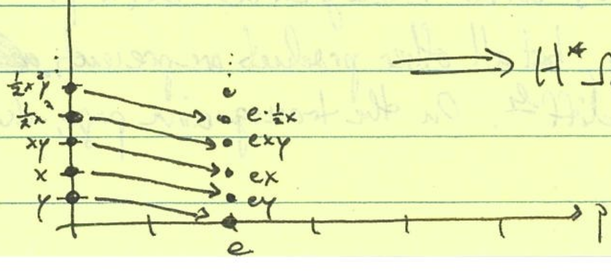
**Ex:** $\Omega S^3 \to * \to S^3$



$\Longrightarrow H^*\Omega S^3 \cong \Gamma[x]$, a "divided power alg." for $|x| = 3$

**Ex:** $\Omega S^2$

$$\begin{tikzcd} \Omega S^2 \arrow[d] \\ * \arrow[d] \\ S^2 \end{tikzcd}$$



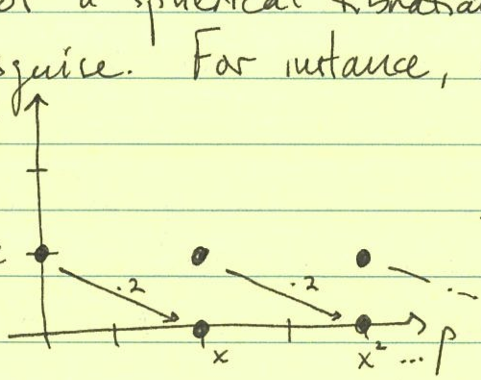
$\Longrightarrow H^*\Omega S^2 \cong \Lambda[y] \otimes \Gamma[x]$ for $|y| = 1$, $|x| = 2$.

(In fact, this dichotomy of $\Omega\Sigma S^n$ and $\Omega S^{2n+1}$ continues.)

__Ex:__ The "Gysin seq$^{\underline{ce}}$" of a spherical fibration is a Serre spectral seq$^{\underline{ce}}$ in disguise. For instance, consider

$$\begin{tikzcd} S^1 \arrow[r, equal] \arrow[d] & S^1 \arrow[d] \\ \mathbb{R}P^\infty \arrow[r] \arrow[d] & * \arrow[d] \\ BS^1 \arrow[r, "\cdot 2"] & BS^1 \end{tikzcd}$$

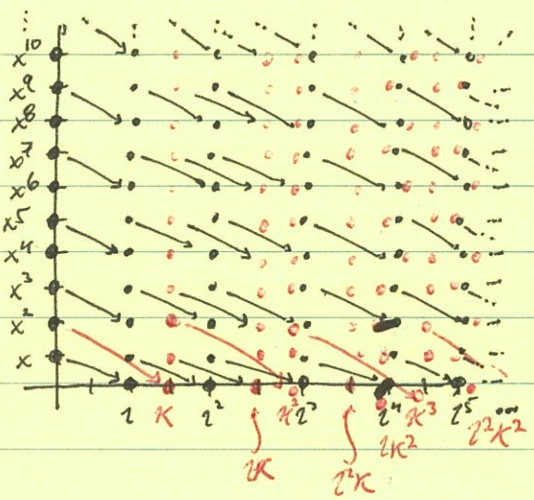
$\rightsquigarrow$



$\Longrightarrow H^*\mathbb{R}P^\infty \cong \mathbb{Z}[x]/(2x)$.

__Rem:__ You'll notice we managed to compute the cohomology of the non-s.c. space $\mathbb{R}P^\infty$ by finding it in a position __other__ than the base. The Serre SS theorem we stated does __not__ apply to $C_2 \to * \to BC_2$.

__Ex:__ $\mathbb{R}P^\infty \to * \to K(\mathbb{F}_2, 2)$ for $H^*(-;\mathbb{F}_2)$



$d(x) = \iota$

$d(x^2) = \kappa$

$d(x^4) = \lambda$

$d(x^8) = \mu$

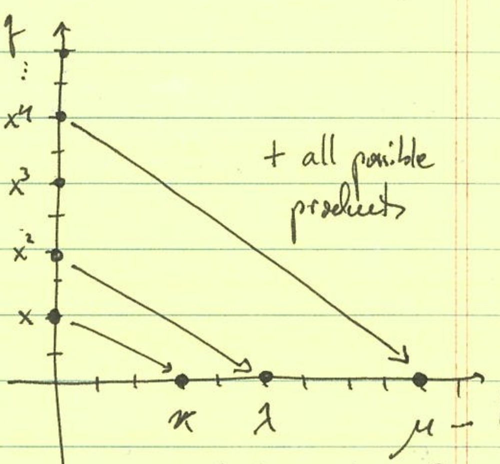
$\vdots \quad \vdots$

$\Downarrow$

$H^*(K(\mathbb{F}_2, 2); \mathbb{F}_2) \cong \mathbb{F}_2[\kappa_j \mid j \geq 0]$

$|\kappa_j| = 2^{j+1} + 1$.

__Ex:__ $K(\mathbb{Z}, 3) \to * \to K(\mathbb{Z}, 2)$ in $H^*(-;\mathbb{F}_2)$



$\Rightarrow H^*(K(\mathbb{Z},3); \mathbb{F}_2) \cong \mathbb{F}_2[\kappa_j \mid j \geq 1]$.

__Ex:__ $K(\mathbb{Z}, 3)$ in $H^*(-;\mathbb{Z})$: Call $H^*\mathbb{C}P^\infty \cong \mathbb{Z}[x]$. Then there's a class $y$ with $d_2 x = y$, and generally $d_2 x^n = n x^{n-1} y$ leaves behind residue $\langle x^{n-1} y \rangle \cong C_n$. At $n=2$, $y^2$ must exist with $d(xy) = y^2$ (and automatically $\langle y^2 \rangle \cong C_2$), so that $\langle x^{n-1} y \rangle = \begin{cases} C_n & \text{if } n \text{ odd}, \\ C_{n/2} & \text{if } n \text{ even}. \end{cases}$ In fact, $y^m$ exist $\forall m$, with ladder of diff$^{\underline{ls}}$. There must exist a class $z_3$ with transgressive $d(x^2 y) = z_3$. There are no mixed products $y^2 z_3$, but all other products are present, and some participate in old short diff$^{\underline{ls}}$. On the transgressive page, also $d(x^5 y) = x^3 z_3, \ldots$

There's no end in sight.

## G-bundles and fiber bundles

A G-bundle is a particular sort of fiber bundle:

Def: A fiber bundle $E \xrightarrow{f} B$ where $G$ acts on $E$ (and trivially on $B$, and the map $p$ is equivariant) is a G-bundle when the identification $\varphi_U : p^{-1}(U) \cong G \times B$ are equivariant and the compatibilities $\varphi_U|_{U \cap V} = \varphi_V|_{U}$ are too.

Rem: If $G$ acts on $F$, one can extract a fiber bundle with fiber $F$ by $E' = (F \times E)/(fg, e) \sim (f, ge)$. Conversely, a fiber bundle with fibers $F$ has an associated $(\mathrm{Aut}\, F)$-bundle.

Rem: $\mathbb{C}$-vector bundles correspond with $GL(\mathbb{C}^n)$-bundles, and $GL(\mathbb{C}^n) = U$

Lem: The assignment $X \longmapsto \{\text{iso}^{\underline{m}}$ classes of G-bundles on $X\}$ satisfies the wedge axiom + Mayer-Victoris. □

Cor: $\exists$ a homotopy type $BG$ representing this functor. □

We would like to understand this homotopy type; today will do it through properties, and later we'll do it through explicit construction.

Lem: Let $E \to B$ be a G-bundle with $E$ $n$-connected. Then the classifying map $B \to BG$ is an $n$-equiv$^{\underline{ce}}$ / the induced natural transformation $[-, B] \to [-, BG] \to k_G(-)$ is an equiv$^{\underline{ce}}$ on complexes of dim $\leq n$. □

Rem: This is 11.27 + 11.35 + HLP in the book. It's not bad, but I don't know how to make it enlightening — and besides, I've promised to give an explicit construction.

Cor: The universal bundle classified by $BG \xrightarrow{id} BG$ has contractible total space, often denoted by $EG$. □

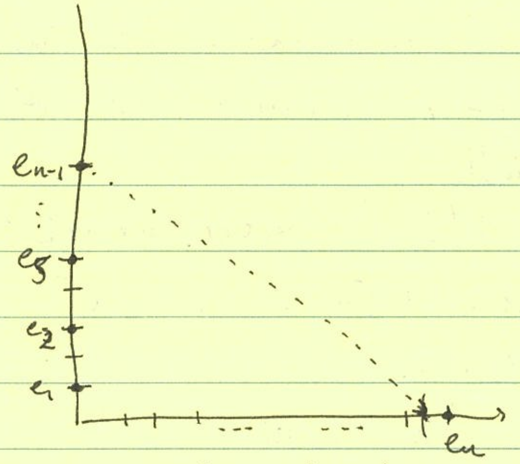
Cor: The lexseq associated to the universal G-bundle on $BG$ shows $\pi_{*+1} BG \cong \pi_* G$. □

~~Ex:~~ Rem: There is a model of $K(G,n)$ which is an actual topological group, so that $BK(G,n) \simeq K(G,n+1)$.

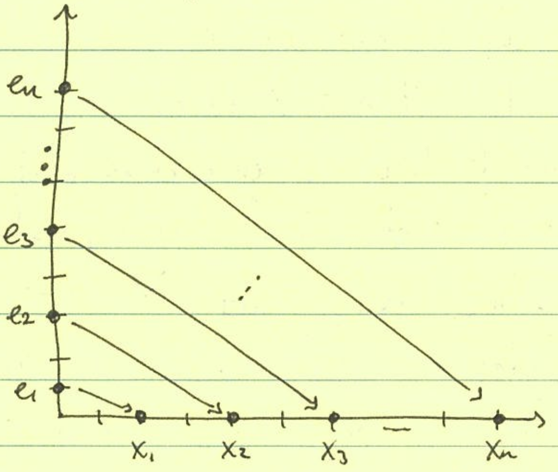
We can also calculate hella cohomology groups.

Thm: $H^*U(n) \cong \Lambda[e_1,\dots,e_n]$. $H^*BU(n) \cong \mathbb{Z}[x_1,\dots,x_n]$.

Pf: We have fiber seq$^{ce}$ $U(n-1) \hookrightarrow U(n) \longrightarrow S^{2n-1}$ and $U(n) \longrightarrow * \longrightarrow BU(n)$ (and, for that matter, $S^{2n-1} \to BU(n-1) \to BU(n)$).



has no room for differentials on indecomposables, hence no diff$^{ls}$ at all. Also no room for multiplicative extensions: no even classes in lower filtration to connect with.



attaches a polynomial class to each of the old exterior classes.

Cor: Associated to each ~~bundle~~ complex vector bundle $V/X$, we have defined a seq$^{ce}$ of classes $c_n(V) \in H^{2n}(X;\mathbb{Z})$, the *Chern classes* of $V$.

Pf: Associated to $V/X$ is a $U(n)$-bundle $\xi/X$, classified by a map $f: X \longrightarrow BU(n)$. This induces a map $f^*: H^*BU(n) \longrightarrow H^*X$, along which we send the classes $x_j$, $j \leq n$. □

# Properties of Chern classes

Thm: For each $U(n)$-bundle $\xi$ over a CW-complex $X$ there are unique elements $c_j(\xi) \in H^{2j}(X)$ depending only on the iso-class of $\xi$, s.t.

(i) For a map $f: Y \to X$, $c_j(f^*\xi) = f^* c_j(\xi)$.

(ii) $c_0(\xi) = 1 \ \forall \xi$. (iii) For $\gamma$ the tautological bundle on $\mathbb{C}P^n$, $c_1(\gamma) = x_1$.

(iv) For $\xi$ a $U(n)$-bundle and $\zeta$ a $U(m)$-bundle $/X$,

$$c_k(\xi \oplus \zeta) = \sum_{i+j=k} (c_i(\xi) \cdot c_j(\zeta)).$$

Lem: For $\xi$ a $\mathbb{C}^n$-bundle on $X$, there exists a space $f: Y \to X$ over $X$ such that (a) $f^*: H^*X \hookrightarrow H^*Y$ and (b) $f^*\xi \cong \xi' \oplus \eta$, where $\eta$ is some line bundle on $Y$.

Pf: Set $Y = \mathbb{P}(\xi)$ to be the fiberwise projectivization of $\xi$; this is a $\mathbb{C}P^{n-1}$-bundle on $X$. The pullback $f^*\xi$ has a natural subbundle $\eta$ ^a 1-dim^ of those vectors in $f^*\xi$ which lie in the line chosen in $\mathbb{P}(\xi)$.

All of our data thus far assemble into the following diagram:

$$\begin{tikzcd}
& \mathbb{C}^\times \arrow[dr, bend left] \arrow[dd, equal] & & & \\
\mathbb{C}^n \arrow[dd] & & \mathbb{C}^n \setminus 0 \arrow[ll, hook'] \arrow[r] \arrow[dd] & \mathbb{C}P^{n-1} \arrow[r] \arrow[dd] & \mathbb{C}P^\infty \arrow[dd, equal] \\
& \mathbb{C}^\times \arrow[dr, bend left] & & & \\
\xi \arrow[d] & & \xi \setminus \text{zero} \arrow[ll, hook'] \arrow[r] \arrow[d] & \mathbb{P}(\xi) \arrow[r] \arrow[d, "f"] & \mathbb{C}P^\infty \arrow[d] \\
X \arrow[rr, equal] & & X \arrow[r, equal] & X \arrow[r] & *
\end{tikzcd}$$

The Serre SS for the red seq^ce^ degenerates, since $H^*\mathbb{C}P^\infty \twoheadrightarrow H^*\mathbb{C}P^{n-1}$.

The edge homomorphism $H^*X \to H^*\mathbb{P}(\xi)$ is thus an inclusion. □

Rem: In $H^*\mathbb{P}(\xi)$, there is a potential mult. extension for $x^{n-1} \cdot x$, i.e., a relation $x^n - b_1 x^{n-1} + b_2 x^{n-2} - \cdots + (-1) b_n = 0$. We will show that these $b$'s model the $c$'s from the theorem + the $x$'s from last time.

Pf of Thm: (i), ~~and~~ (ii), and (iii) are automatic for the $b$'s. To get unicity, apply the construction twice: first to split $\xi$ into $\xi' \oplus \eta$, then to compute the Chern classes of $\xi'$ and $\eta$, and find $c_j(\xi) = c_j(\xi') + c_1(\eta) c_{j-1}(\xi')$.

To get the claim about sums of bundles, note that $\mathbb{P}(\eta)$, $\mathbb{P}(\zeta)$ are subspaces of $\mathbb{P}(\eta \oplus \zeta)$ s.t. $\mathbb{P}(\zeta)$ is a deformation retract of $\mathbb{P}(\eta \oplus \zeta) \setminus \mathbb{P}(\zeta)$ and vice versa. Form the sums $b_\eta = \sum_{j=0}^{m} (-1)^j b_j(\eta) y^{m-j}$ and $b_\zeta = \sum_{j=0}^{n} (-1)^j b_j(\zeta) y^{n-j}$ as elements of $H^* \mathbb{P}(\eta \oplus \zeta)$. Then $b_\eta|_{\mathbb{P}(\eta)} = 0$ and $b_\zeta|_{\mathbb{P}(\zeta)} = 0$, so a Mayer-Vietoris argument says $b_\eta \cdot b_\zeta = 0$ in $\mathbb{P}(\eta \oplus \zeta)$. But $b_{\eta \oplus \zeta}$ is the unique monic poly$^{\underline{l}}$ with this property (of deg $n+m$). □

To get the claim about $b_j$'s and the $x_j \in H^{2j} BU(n)$, consider the maps $(\mathbb{C}P^\infty)^{\times n} \to BU(n)$ classifying the universal $\mathbb{C}^n$-bundle with a decomposition into $n$ lines. This participates in a map of les seq$^{\underline{s}}$:

$$\begin{tikzcd}
 & x_1, \dots, x_n & x_1, \dots, x_n & x_1, \dots, x_{n-1} & \\
H^{*+2n-1} BU(n-1) \arrow[r, "0"] \arrow[d] & H^* BU(n) \arrow[r, hook, "\cdot x_n"] \arrow[d] & H^{*+2n} BU(n) \arrow[r, "①"] \arrow[d] & H^{*+2n} BU(n-1) \arrow[r, "0"] \arrow[d, "☆②"] & H^{*+1} BU(n) \arrow[d] \\
0 \arrow[r] & H^* (\mathbb{C}P^\infty)^{\times n} \arrow[r] & H^{*+2n} (\mathbb{C}P^\infty)^{\times n} \arrow[r, "☆①"] & H^{*+2n} (\mathbb{C}P^\infty)^{\times (n-1)} \arrow[r] & 0 \\
 & \pi_1^* x_1, \dots, \pi_n^* x_{n-1} & \pi_1^* x_1, \dots, \pi_n^* x_1 & \pi_1^* x_1, \dots, \pi_{n-1}^* x_1 &
\end{tikzcd}$$

The top is the Gysin seq$^{\underline{ce}}$. Assuming the theorem at $n-1$, the map ☆ sends $x_j$ to $b_j(\eta^{\oplus(n-1)}) = \sigma_j(\pi_1^* x_1, \dots, \pi_{n-1}^* x_1)$, an elementary symm. f$^{\underline{n}}$. In general, the $x_j$ land in the symm. f$^{\underline{ns}}$, and $x_n$ lands in the kernel of ☆, hence is a constant multiple of $\sigma_n$. Actually, the vertical maps are all ring maps, which puts a huge restriction on their behavior: $x_n$ must be sent to $\sigma_n$ on the nose. □

## The bar construction

For $G$ a finite group, there is an especially useful model for the classifying space for $G$-bundles: the bar complex.

**Def:** For $\mathcal{C}$ a category, we construct its nerve $\swarrow N(\mathcal{C})$ as a simplicial set with 0-simplices = ob $\mathcal{C}$, 1-simplices = arr $\mathcal{C}$, 2-simplices = $\begin{tikzcd}[row sep=small] & \bullet y \arrow[dr, "g"] & \\ x\bullet \arrow[ur, "f"] \arrow[rr, "gf"'] & & \bullet z \end{tikzcd}$ commuting triangles, 3-simplices = $\begin{tikzcd}[row sep=small] & & z \arrow[dr, "h"] & \\ & y \arrow[ur, "g"] \arrow[rr, "hg"] & & a \\ x \arrow[ur, "f"] \arrow[uurr, "gf"] \arrow[urrr, "hgf"'] & & & \end{tikzcd}$, ... .

**Rem:** This construction translates functors to ct maps + natural transformations to homotopies of maps. (In some sense, this is where nat. trans. came from.)

**Ex:** For $G$ a group, we describe two categories:

$G/\!/G$ has objects $g \in G$ + maps $g \xrightarrow{h} gh$.

$*/\!/G$ has one object $*$ + maps $* \xrightarrow{h} *$.

**Lem:** $G/\!/G$ is contractible. (Any map in from $*$ is f.f. + essentially surj.) $\square$

**Rem:** The $G$-action on $G/\!/G$ is free.

**Cor:** $N(G/\!/G) \to N(*/\!/G)$ models $EG \longrightarrow BG$. $\square$

"The Grothendieck construction"

**Rem:** This is a more conceptual statement than you might think. There are equiv^ces^

$$\begin{tikzcd} G/\!/G \arrow[r] \arrow[d, "\simeq"'] & */\!/G \arrow[d, "\simeq"'] \\ \left\{\begin{smallmatrix} G\text{-torsors w/} \\ \text{a triv}^{\text{n}} \end{smallmatrix}\right\} \arrow[r, two heads] & \{G\text{-torsors}\} \end{tikzcd}$$

and a map $X \longrightarrow N\{G\text{-torsors}\}$, for $X$ a simplicial set assigns each point in $X$ to a $G$-torsor, each path to a map of torsors, ... . This sounds like it's building a $G$-bundle on $X$ by specifying the fiber.

This very concrete model for $BG$ has one really excellent feature: it has a naturally occurring skeletal filtration w/ identifiable quotients:

$$\begin{tikzcd}[column sep=small] * = BG^{(0)} \arrow[r] \arrow[d, equal] & BG^{(1)} \arrow[r] \arrow[d] & BG^{(2)} \arrow[r] \arrow[d] & BG^{(3)} \arrow[r] \arrow[d] & \cdots \arrow[r] & BG^{(n-1)} \arrow[r] \arrow[d] & BG^{(n)} \arrow[r] \arrow[d] & \cdots \\ * & \Sigma G & (\Sigma G)^{\wedge 2} & (\Sigma G)^{\wedge 3} & \cdots & (\Sigma G)^{\wedge (n-1)} & (\Sigma G)^{\wedge n} & \cdots \end{tikzcd}$$

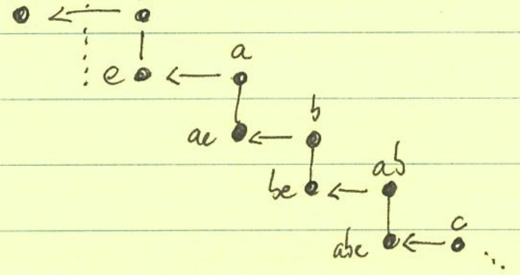
.

If $h$ is a homology theory with Künneth iso$^{\underline{ms}}$, this gives a SS
$$E^1_{*,*} = (\tilde{h}_* \Sigma G)^{\otimes *} \Rightarrow h_* BG.$$
More than this, the $d^1_1$-diff$^{\underline{l}}$ is then identifiable: $d_1(g_1 \otimes \cdots \otimes g_n) = \Sigma_{j=2}^{n} (g_1 \otimes \cdots \otimes g_{j-1} g_j \otimes \cdots \otimes g_n)$.
This is a standard resolution appearing in homological algebra: $E^2_{*,*} = \mathrm{Tor}^{h_* G}_{*,*}(h_*, h_*)$.

Rem: This is a very common situation: some "fully derived" construction appearing in homotopy theory has behavior mediated by homological algebra and a spectral sequence. "BG is some kind of $* \times_G *$."

Tor algebras are actually remarkably computable: there is an algorithm, due to Tate, which forms a resolution of $h_*$ by a DGA which is levelwise $(h_* G)$-free.

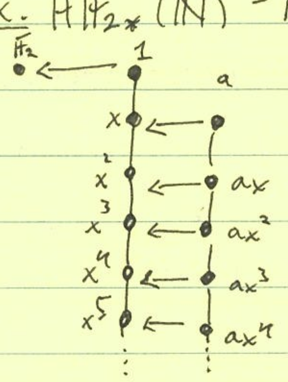
~~Ex: $h_*(\mathbb{Z}/2) = h_*\{[0], [1]-[0]\}$~~ Ex: $H\mathbb{F}_{2*}(\mathbb{Z}/2) = \mathbb{F}_2\{[0], [1]-[0]\} \big/ ([1]-[0])^2 = 0$.



$$\xrightarrow{-\otimes \mathbb{F}_2} \mathbb{F}_2[\overset{\overset{1}{\downarrow}}{a}, \overset{\overset{2}{\downarrow}}{b}, \overset{\overset{4}{\downarrow}}{c}, \overset{\cdots 2^n \cdots}{\ldots}] \cong \mathrm{Tor}^{\Lambda[e]}_{*,*}(\mathbb{F}_2, \mathbb{F}_2).$$

This is a/k/a a divided polynomial algebra/$\mathbb{F}_2$.

Ex: $H\mathbb{F}_{2*}(\mathbb{N}) = \mathbb{F}_2\{[0], [1], [2], \ldots\} \cong \mathbb{F}_2[x]$.



$$\xrightarrow{-\otimes \mathbb{F}_2} \mathbb{F}_2[a] \big/ a^2 \cong \mathrm{Tor}^{\mathbb{F}_2[x]}_{*,*}(\mathbb{F}_2, \mathbb{F}_2).$$

Rem: There is a really slick proof of complex Bott periodicity that uses these nerve constructions. It's on your homework.

## The Steenrod algebra: calculation

Recall from a while ago that we were interested in exhaustively computing the set of natural transformations $H^n(-;A) \longrightarrow H^{n+2}(-;B)$. We will do this today in the case of $A=B=\mathbb{F}_2$, using the bar spectral seq$^{\underline{ce}}$.

Our method is inductive, and it rests on two key observations:

Lem: The bar construction applied to a topological group, like $K(\mathbb{F}_2, n)$, still produces a delooping, like $K(\mathbb{F}_2, n+1)$. ☐

Lem: The pairings $K(\mathbb{F}_2, n) \times K(\mathbb{F}_2, 1) \xrightarrow{\smile} K(\mathbb{F}_2, n+1)$ induces a pairing of spectral seq$^{\underline{ces}}$

$$\begin{tikzcd}
\mathrm{Tor}^{H_*K(\mathbb{F}_2,n)}_{*,*} \otimes H_* \mathbb{RP}^\infty \arrow[r, "\circ"] \arrow[d, Rightarrow] & \mathrm{Tor}^{H_*K(\mathbb{F}_2,n+1)}_{*,*} \arrow[d, Rightarrow] \\
H_*K(\mathbb{F}_2, n+1) \otimes H_* \mathbb{RP}^\infty \arrow[r, "\overset{\smile}{\bullet}"] & H_*K(\mathbb{F}_2, n+2),
\end{tikzcd}$$

which satisfies $d(x \circ y) = (dx) \smile y$. ☐

Base case: We know $H_* \mathbb{RP}^\infty$ has one class in every degree. The bar spectral seq$^{\underline{ce}}$ takes as input $E^2_{*,*} = \mathrm{Tor}^{H_*(\mathbb{Z}/2)}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \Gamma[a]$, which has one class in every degree. There can therefore be no diff$^{\underline{ls}}$. We write $\Gamma[a] \cong \mathbb{F}_2[a_{(0)}, a_{(1)}, a_{(2)}, \ldots]/(a_{(j)}^2 = 0)$ for the algebra gens.

Induction: We claim that this is true generically:

$H_* K(\mathbb{F}_2, n) \cong \mathbb{F}_2[a_{(j_1)} \smile \cdots \smile a_{(j_n)}]/(\text{squares})$. This is a tensor product of exterior algebras, so the Künneth $\text{th}^{\underline{m}}$ for $\mathbb{F}_2$-algebras gives

$\mathrm{Tor}^{H_*K(\mathbb{F}_2,n)}_{*,*} \cong \bigotimes_J \mathrm{Tor}^{\mathbb{F}_2[a_{(j_1)} \smile \cdots \smile a_{(j_n)}]/\text{square}}_{*,*} \cong \bigotimes_J \Gamma[a_{(j_1)} \smile \cdots \smile a_{(j_n)}]$.

Wilson 8.16 ⟶ Claim: $(a_{(J)})_{(k)} \equiv a_{(J)} \bullet\smile a_{(k)}$ mod decomposables.

Consequence: There are no diff$^{\underline{ls}}$, since $H_* \mathbb{RP}^\infty$ had none. ☐

There are a lot of consequences to draw from this. For instance, we can calculate $H^* K(\mathbb{F}_2, n)$ by taking the $\mathbb{F}_2$-linear dual. The fastest way to name the outcome is that the dual of $\Gamma[-]$ is $\mathbb{F}_2[-]$.

We also get *stable* information out of this. From our def$^{\underline{n}}$ of $H\mathbb{F}_2 \wedge H\mathbb{F}_2$, we have $H\mathbb{F}_2 \wedge H\mathbb{F}_2 \simeq H\mathbb{F}_2 \wedge (\varinjlim \Sigma^{-n}_\infty K(\mathbb{F}_2, n)) \simeq \varinjlim (H\mathbb{F}_2 \wedge \Sigma^{-n}_\infty K(\mathbb{F}_2, n))$, so that $(H\mathbb{F}_2)_m H\mathbb{F}_2 = \varinjlim_{n \to \infty} H\mathbb{F}_{2\,m-n}(K(\mathbb{F}_2, n))$, so our calculation gives us access to these groups if we can describe the maps in the colimit. These turn out to be $- \smile a_{(0)}$:

$$\begin{tikzcd} S^1 \wedge K(\mathbb{F}_2, n) \arrow[r] \arrow[dr, "a_{(0)} \times 1"'] & K(\mathbb{F}_2, n+1) \\ & K(\mathbb{F}_2, 1) \wedge K(\mathbb{F}_2, n) \arrow[u, "\smile"'] \end{tikzcd}$$ .

<u>Cor</u>: $H\mathbb{F}_{2*} H\mathbb{F}_2 = \mathbb{F}_2[\xi_1, \xi_2, \ldots, \xi_n, \ldots]$, where $|\xi_n| = 2^n - 1$ is rep$^{\underline{d}}$ by $a_{(n)} \in (H\mathbb{F}_2)_{2^n-1}(\Sigma^{-1}_\infty K(\mathbb{F}_2, 1))$. ☐

The diagonal on $H\mathbb{F}_{2*} H\mathbb{F}_2$ is given by $\Delta \xi_n = \sum_{j=0}^{n} \xi_j \otimes \xi_{n-j}^{2^j}$.

Lots more formulas like this can be read off. Maybe on your homework? ☺

<u>Cor</u>: The primitive elements of this algebra are $\xi_1^{2^j}$. ☐

<u>Cor</u>: The dual algebra, $H\mathbb{F}_2^* H\mathbb{F}_2$, is generated by elements $Sq^{2^j}$ dual to $\xi_1^{2^j}$. It is <u>non-commutative</u> with diagonal $\Delta Sq^n = \sum_{n_1+n_2=n} Sq^{n_1} \otimes Sq^{n_2}$.

The homology version of this is $\mathcal{A}_*$, the <u>dual Steenrod algebra</u>. The cohomology version is $\mathcal{A}^*$, the <u>Steenrod algebra</u>. The space-level versions are called the <u>unstable (dual) Steenrod algebra</u>.

A lot about these operations can be computed in the universal case. For instance, $\Delta(x^2)^* = 1 | (x^2)^* + \xi_1 | (x)^* \in \mathcal{A}_* \otimes \tilde{H}_* \mathbb{R}P^\infty$ says $Sq^0(x^4) = x^2$ and $Sq^1(x) = x^2$. In fact, we have

⓪ $Sq^0(x) = x$ in general. ① $Sq^{>|x|}(x) = 0$.

② $Sq^{|x|}(x) = x^2$ in general. ③ $Sq^n(x+y) = Sq^n x + Sq^n y$.

④ $Sq^n(xy) = \sum_{n_1+n_2=n} Sq^{n_1}(x) \cdot Sq^{n_2}(y)$.

⑤ "The Adem relations." These are summarized by

(i) $Sq^{2n-1} Sq^n = 0$, (ii) $d(Sq^n) = Sq^{n-1}$ extends to a derivation.

<u>Consequence</u>: $0 = d^3(Sq^5 Sq^3) = d(Sq^3 Sq^3 + Sq^5 Sq^1) = Sq^2 Sq^3 + Sq^3 Sq^2 + Sq^4 Sq^1 + Sq^5 Sq^0$.

## The Steenrod algebra: interaction w/ SSS

Last time, we gave a computation of $A_*$ and $A^*$, the $\mathbb{F}_2$-Steenrod algebra + its dual, using the bar spectral seq$^{\underline{ce}}$. Separately, we computed a few of the SSSes for E-M spaces as toy examples. Today we systematize this + note an exciting consequence: Kudo's theorem.

When we were re/proving Freudenthal's theorem, we made the following construction: $d_n : H^0_0(B; H^n F) \longrightarrow H^{n+1}(B; H^0 F)$ we reinterpreted as the maps $(B, \cancel{B} b_0) \xleftarrow{p} (E, F) \xrightarrow{j} (E \cup CF, CF) \xrightarrow{i} (\Sigma F, *)$, and we said that $f \in H^n F$ <u>transgresses</u> to $b \in H^{n+1} B$ if $j^* i^* f = p^* b$.

Kudo transgression

<u>Lem</u>: If $f$ transgresses to $b$, then $Sq^m f$ transgresses to $Sq^m b$.

Pf: $j^* i^* Sq^m f = Sq^m j^* i^* f = Sq^m p^* b = p^* Sq^m b$, just using naturality. $\square$

This will turn out to be a powerful computational tool, but it will also just be useful for naming things.

<u>Thm</u>: $H\mathbb{F}_2^* K(\mathbb{F}_2, q) = \mathbb{F}_2[Sq^I \iota_q \mid I_j \geq 2(I_{j+1}),\ 2I_1 - I_+ < q]$.

Pf: This is mostly a matter of organization. The map $\Sigma K(\mathbb{F}_2, q) \to K(\mathbb{F}_2, q+1)$ guarantees that the fundamental class transgresses to the fundamental class in the SSS for the fibration $K(\mathbb{F}_2, q) \to * \to K(\mathbb{F}_2, q+1)$. Each polynomial generator $Sq^I \iota_q$ is thus sent to $Sq^I \iota_{q+1}$ by the diff$^{\underline{l}}$. In fact, this is true of the squares, since e.g. $(Sq^I \iota_q)^2 = Sq^{I_+ + q} Sq^I \iota_q$. The two conditions are visibly true for this new class, and conversely all such classes arise. $\square$

<u>Thm</u>: $H\mathbb{F}_2^* K(\mathbb{Z}, q) = \mathbb{F}_2[Sq^I \iota_q \mid \cdots$, <u>and</u> $I_{\text{final}} \neq 1]$.

Pf: This is entirely similar, with a different base case. $\square$

There is a useful trick here for killing classes in the cohomology of spaces. It will seem a little "out of left field", but we will use it to great effect in a couple of lectures.

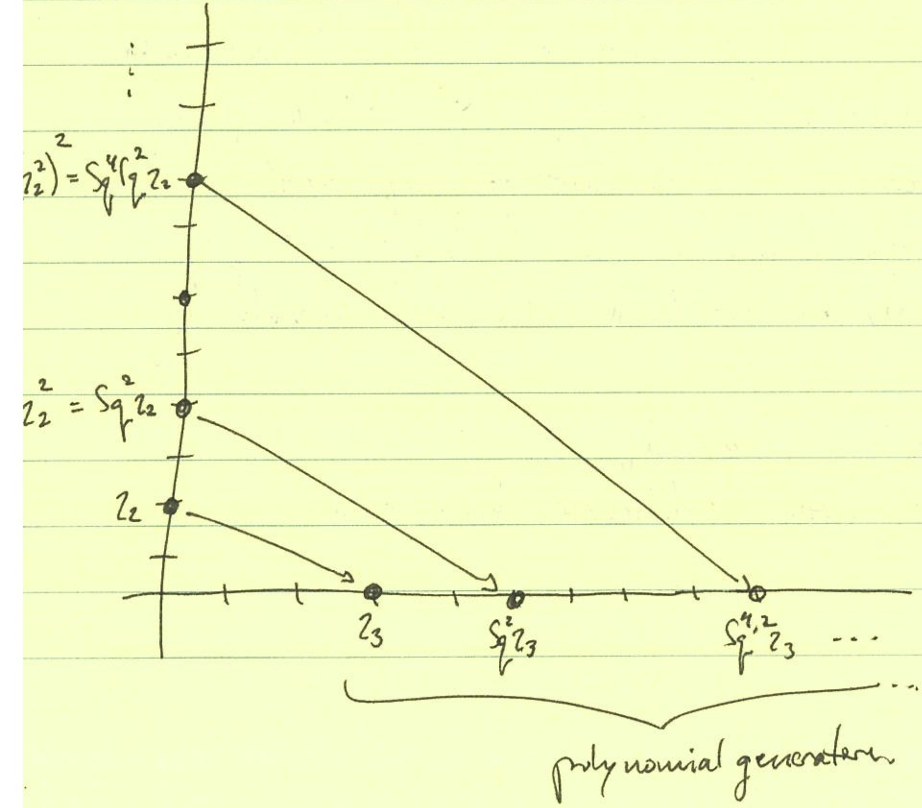
Consider a cohomology class $\omega \in H^n(X;A)$, represented by a map $X \xrightarrow{\omega} K(A,n)$. In the spectral seq$^{\underline{ce}}$ for $K(A,n-1) \to * \to K(A,n)$, we know that the fundamental class transgresses. Now consider the pulled back fibration, $P_\omega$. The fundamental class also transgresses in this spectral seq$^{\underline{ce}}$, because it does in the universal case + because $\iota$ pulls back along $\omega$ to $\omega \neq 0$.

$$\begin{tikzcd}
K(A,n-1) \arrow[r, equal] \arrow[d] & K(A,n-1) \arrow[d] \\
P_\omega \arrow[r] \arrow[d] & * \arrow[d] \\
X \arrow[r, "\omega"] & K(A,n)
\end{tikzcd}$$

□

<u>Cor:</u> $\omega$ does not survive the spectral seq$^{\underline{ce}}$.

This is kind of a wild op$^{\underline{n}}$ if you actually write it out. The main point is that Kudo still gives you a handle on what's going on.



## Serre classes

The idea is to build a version of homotopy theory that only thinks about a particular prime. We already built a version of homotopy theory that only cares about homotopy type — this is an elaboration of that.

Sometimes we just ask for $A \otimes A'$ and $\mathrm{Tor}(A, A')$ to lie in $\mathcal{C}$.

**Def:** A class of abelian gp^s is a coll^n $\mathcal{C}$ such that …

(i) For $0 \to A' \to A \to A'' \to 0$, $A \in \mathcal{C} \iff A', A'' \in \mathcal{C}$. "closed under +, −"

(ii) For $A \in \mathcal{C}$ and $B$ a gp, $A \otimes B \in \mathcal{C}$. "closed under ·, "ideal""

(iii) For $A \in \mathcal{C}$, $H_{*>0}(BA; \mathbb{Z}) \in \mathcal{C}$.

**Ex:** $\mathcal{C}_p$: abelian torsion gp^s of finite exponent + every order of every elt is prime t

**Big idea:** Algebra works "up to $\mathcal{C}$". a $\mathcal{C}$-mono has kernel in $\mathcal{C}$. A $\mathcal{C}$-epi has cokernel in $\mathcal{C}$. A $\mathcal{C}$-iso has both, and two gp^s are $\mathcal{C}$-iso^c if they're connected by a zig-zag of $\mathcal{C}$-iso^s.

**Bigger idea:** Homological algebra also works "up to $\mathcal{C}$". There are notions of $\mathcal{C}$-exactness, the 5-lemma up to $\mathcal{C}$, the snake lemma up to $\mathcal{C}$, … .

**Biggest idea:** Homotopy theory (of simply connected spaces) works "up to $\mathcal{C}$".

(Hurewicz) **Thm:** For $X$ a s.c. space, if $\pi_{<n} X \in \mathcal{C}$ then $H_{<n} X \in \mathcal{C}$ and $\pi_n X \to H_n X$ is a $\mathcal{C}$-iso^m.

(Whitehead) **Thm:** For $f: X \to Y$ a map of s.c. spaces which is an iso^m on $\pi_2$, $f$ induces a $\mathcal{C}$-iso^m on $\pi_{<n}$ and a $\mathcal{C}$-epi on $\pi_n$ iff $f$ induces a $\mathcal{C}$-iso^m on $H_{<n}$ and a $\mathcal{C}$-epi on $H_n$.

(Approx^n) **Thm:** Let $X, Y$ be s.c. spaces and let $f: Y \to X$ have $\pi_2$ epi. TFAE:

(i) $H^{<n}(X; \mathbb{Z}/p) \to H^{\leq n}(Y; \mathbb{Z}/p)$ iso + $H^n(X; \mathbb{Z}/p) \to H^n(Y; \mathbb{Z}/p)$ mono.

duality

(ii) $H_{<n}(Y; \mathbb{Z}/p) \to H_{<n}(X; \mathbb{Z}/p)$ iso + $H_n(X; \mathbb{Z}/p) \to H_n(Y; \mathbb{Z}/p)$ epi.

lexseq of a pair

(iii) $H_{\leq n}(X, Y; \mathbb{Z}/p) = 0 \overset{\text{UCT}}{\longleftrightarrow}$ (iv) $H_{\leq n}(X, A; \mathbb{Z}) \in \mathcal{C}_p$.

rel. Hur.

(v) $\pi_{\leq n}(X, Y) \in \mathcal{C}$. $\pi_*$ lexseq (vi) $\pi_{<n} Y \to \pi_{<n} X$ is $\mathcal{C}_p$-iso + $\pi_n Y \to \pi_n X$ is a $\mathcal{C}_p$-epi.

pf in reverse

(vii) $\pi_{<n} X$ and $\pi_{<n} Y$ have iso^c $p$-components.

Pf of Ex (iii): The other properties of $\mathcal{C}_p$ are obvious, but $H_*BA$ is not. Thankfully, we have the bar spectral seq$^{\underline{ce}}$, computing $H_*BA \Leftarrow \mathrm{Tor}^{H_*A}_{*,*}(\mathbb{Z},\mathbb{Z})$. We know also that for $q$ the exponent of $A$, the map $BA \xrightarrow{\Delta} BA^{\times q} \xrightarrow{\mu} BA$ is null-homotopic, so all classes are $*$-nilpotent of order $q$. Distributivity finishes the proof. □

Lem: For $f: A \to B$ a $\mathcal{C}_p$-iso$^{\underline{m}}$ of f.g. grps, $A$ and $B$ have iso$^{\underline{c}}$ $p$-components.

Pf: Let ${}_pA$ denote the subgp of torsion elements prime to $p$, so that we want $A/{}_pA \cong B/{}_pB$. In

$$\begin{tikzcd} 0 \arrow[r] & {}_pA \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "f"] & A/{}_pA \arrow[r] \arrow[d, "\hat{f}"] & 0 \\ 0 \arrow[r] & {}_pB \arrow[r] & B \arrow[r] & B/{}_pB \arrow[r] & 0, \end{tikzcd}$$

~~everything~~ $f$ is a $\mathcal{C}_p$-iso$^{\underline{m}}$, hence $\hat{f}$ is a $\mathcal{C}_p$-iso$^{\underline{m}}$ by the $\mathcal{C}_p$-5-lemma. This plus FTFGAG $\Rightarrow$ $\hat{f}$ is a mono, $\hat{f}$ induces an iso$^{\underline{m}}$ of torsion subg$^{\underline{ps}}$, and its image is a subgp of maximum rank. □

## "Serre's method"

Our goal is to use the techniques built so far to start computing homotopy groups. The idea is to run the construction of E-M spaces (or, more generally, the proof of Brown representability) "in reverse": we start with an E-M space as a model of $\pi_n$ of an $(n-1)$-connected space + correct its co/homology to match, so that the fiber becomes more+more connected.

We will want the following recorded:

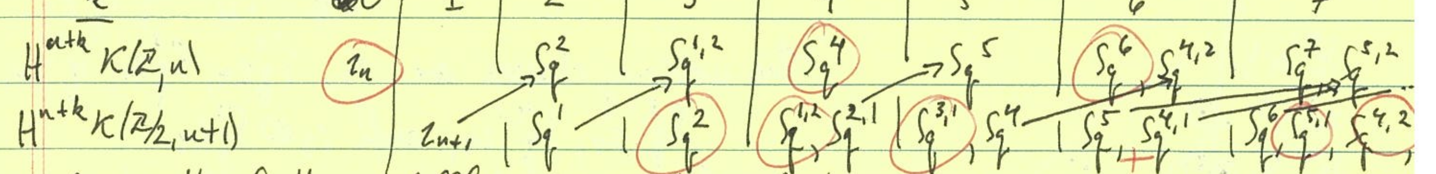
$$H\mathbb{F}_2^* K(\mathbb{Z}/2, q) = \mathbb{F}_2[Sq^I \iota_q \mid I_j \geq 2(I_{j+1}), \text{ and } 2I_1 - I_+ < q],$$

$$H\mathbb{F}_2^* K(\mathbb{Z}, q) = \mathbb{F}_2[Sq^I \iota_q \mid I_j \underset{①}{\geq} 2(I_{j+1}), \ 2I_1 \underset{②}{-} I_+ < q, \text{ and } I_{\underset{③}{\text{last}}} \neq 1].$$

Suppose $n \geq \underline{1}$; we've then shown $\pi_n S^n \cong \mathbb{Z}$. So, the map $S^n \to K(\mathbb{Z}, n)$ witnesses $K(\mathbb{Z}, n) \simeq S^n[0, n]$, a down-ward Postnikov truncation. However, their cohomology is different: $H\mathbb{F}_2^{n+2} K(\mathbb{Z}, n) \ni Sq^2 \iota_n$, but $H\mathbb{F}_2^{n+2} S^n = 0$. We aim to correct this by "fibering out" this class, as indicated at right. The new space $S^n[0, n+1]$ has cohomology presented by a Serre spectral seq$^{\underline{ce}}$ w/ lots of diff$^{\underline{ls}}$.

$$\begin{tikzcd}
& K(\mathbb{Z}/2, n+1) \arrow[r, equal] \arrow[d] & K(\mathbb{Z}/2, n+1) \arrow[d] \\
& S^n[0, n+1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d] \\
S^n \arrow[r] \arrow[ur, dashed, "\exists"] & K(\mathbb{Z}, n) \arrow[r] & K(\mathbb{Z}/2, n+2)
\end{tikzcd}$$

<u>Assumption</u>: For $n \gg 0$, we don't have to worry about ② or product terms.



| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $H^{n+k} K(\mathbb{Z}, n)$ | $\iota_n$ | | $Sq^2$ | $Sq^{1,2}$ | $Sq^4$ | $Sq^5$ | $Sq^6$, $Sq^{4,2}$ | $Sq^7$, $Sq^{5,2}$ … |
| $H^{n+k} K(\mathbb{Z}/2, n+1)$ | | $\iota_{n+1}$ | $Sq^1$ | $Sq^2$ | $Sq^{1,2}$, $Sq^{2,1}$ | $Sq^{3,1}$, $Sq^4$ | $Sq^5$, $Sq^{4,1}$ | $Sq^6$, $Sq^{5,1}$, $Sq^{4,2}$ |

where all of these diff$^{\underline{ls}}$ are given by Kudo transgression. The classes highlighted in red are those that survive to the $E_\infty$-page —— which, again, does not match $H^* S^n$, with first aberration in degree $n+3$. As before, we "fiber out" this bad class, giving us a new spectral seq$^{\underline{ce}}$ to worry about.

$$\begin{tikzcd}
& K(\mathbb{Z}/2, n+2) \arrow[r, equal] \arrow[d] & K(\mathbb{Z}/2, n+2) \arrow[d] \\
& S^n[0, n+2] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d] \\
S^n \arrow[r] \arrow[ur, dashed, "\exists"] & S^n[0, n+1] \arrow[r] & K(\mathbb{Z}/2, n+3)
\end{tikzcd}$$

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $H^{n+k} S^n[0,n+1]$ | $\iota_n$ | | | $Sq^2 \iota_{n+1}$ | $Sq^4 \iota_n$, $Sq^3 \iota_{n+1}$ | $Sq^{3,1} \iota_{n+1}$ | $Sq^6 \iota_n$, $(Sq^5 + Sq^{4,1}) \iota_{n+1}$ | $Sq^7 \iota_n$, $Sq^{5,1} \iota_{n+1}$, $Sq^{4,2} \iota_n$ |
| $H^{n+k} K(\mathbb{Z}/2, n+2)$ | | | $\iota_{n+2}$ | $Sq^1 \iota_{n+2}$ | $Sq^2 \iota_{n+2}$ | $Sq^3$, $Sq^{2,1}$ | $Sq^{3,1}$, $Sq^4$ | $Sq^5 + Sq^{4,1}$ |

We would like to do this again, but there is a problem: the class $Sq^4 \iota_n$ has no $Sq^1$, so fibering it out with a $K(\mathbb{Z}/2, n+3)$ will not kill it — $Sq^1 \iota_{n+3}$ will take its place. It turns out (Ch 11-12 of Mosher + Tangora) that $K(\mathbb{Z}/8, n+3)$ is the correct choice, but we have not set things up so as to make this obvious. ☹

If you believe this, then this gives

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $H^{n+k} S^n[0,n+2]$ | $\iota_n$ | | | | $Sq^4 \iota_n$ | $Sq^3 \iota_{n+2}$ | $Sq^6 \iota_n$ | $Sq^7 \iota_n$, $(Sq^5 + Sq^{4,1}) \iota_{n+2}$ |
| $H^{n+k} K(\mathbb{Z}/8, n+3)$ | | | | $\iota_{n+3}$ | $\beta_3 \iota_{n+3}$ | $Sq^2 \iota_{n+3}$ | $Sq^3 \iota_{n+3}$, $Sq^2 \beta_3 \iota_{n+3}$ | $Sq^4 \iota_{n+3}$, $Sq^3 \beta_3 \iota_{n+3}$ |

and we conclude that the next class lies in degree 7.

We have thus learned: $\pi_{1+n} S^n \cong \mathbb{Z}/2$, $\pi_{2+n} S^n \cong \mathbb{Z}/2$, $\pi_{3+n} S^n \cong \mathbb{Z}/8$, $\pi_{4+n} S^n \cong 0$, $\pi_{5+n} S^n \cong 0$, $\pi_{6+n} S^n \not\cong 0$

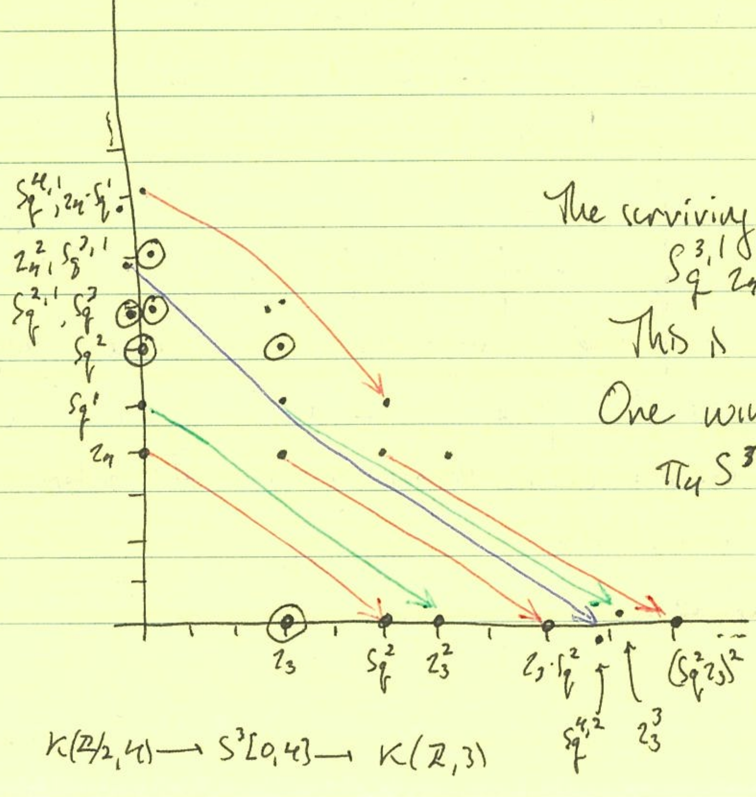
We could, presumably, keep going.

Rem: There is an indeterminacy at $k = 14$ which cannot be resolved w/ methods here. We will meet this again.

Rem: For contrast, here's a computation in very low degrees ($\leq 9$):

~~$d_8$~~ $d_4 \iota_4 = Sq^2 \iota_3$. $d_5 Sq^1 \iota_4 = \iota_3^2$.

$d_8 Sq^4 \iota_4 = Sq^4 Sq^2 \iota_3$.



The surviving classes are $\iota_3$, $Sq^2 \iota_4$, $Sq^3 \iota_4$, $Sq^{2,1} \iota_4$, $Sq^{3,1} \iota_4$, $Sq^{4,1} \iota_4$, and $\iota_3 \otimes Sq^2 \iota_4$.

This is considerably more intricate.

One winds up computing $\pi_3 S^3 \cong \mathbb{Z}$, $\pi_4 S^3 \cong \mathbb{Z}/2$, $\pi_5 S^3 \cong \mathbb{Z}/2$, $\pi_6 S^3 \cong \mathbb{Z}/4$.

# The Adams Spectral Seq^ce

Our goal is to repackage all of the "$n \gg 0$" content of the previous lecture into a single beautiful machine. In order to do this most handily, we return to the setting of stable homotopy.

The basic idea is simple: try to strip a spectrum of its homotopy by iteratively applying the Hurewicz map.

Lem: Take $X$ to be $(n-1)$-conn^d with $\pi_n X$ finite exponent + 2-torsion. Let $X' = \mathrm{fib}(X = \mathbb{S} \wedge X \longrightarrow H\mathbb{F}_2 \wedge X)$. Then $\pi_n X' \subsetneq \pi_n X$.

Pf: Hurewicz says that $\pi_{<n}(H\mathbb{F}_2 \wedge X) = 0$, $\mathrm{im}(\pi_n X \longrightarrow \pi_n H\mathbb{F}_2 \wedge X) = (\pi_n X)/2$, and $\pi_{n+1} X \twoheadrightarrow H_n X$. This gives $X'$ $(n-1)$-conn^d + $\pi_n X' \subsetneq \pi_n X$. □

Thm: Let $\overline{H} \longrightarrow \mathbb{S} \xrightarrow{\eta} H$ be the fiber of the unit map. For $X$ a connective spectrum w/ $\pi_* X$ 2-torsion + degreewise finite exponent,

$$\begin{tikzcd} \mathbb{S} \wedge X \arrow[d] & \overline{H} \wedge X \arrow[l] \arrow[d] & \overline{H}^{\wedge 2} \wedge X \arrow[l] \arrow[d] & \cdots \arrow[l] \\ H \wedge X & H \wedge \overline{H} \wedge X & H \wedge \overline{H}^{\wedge 2} \wedge X & \end{tikzcd}$$

has contractible limit + a strongly convergent SS to $\pi_* X$.

Pf: This spectral seq^ce is exactly the Lem, repeatedly applied. (You also need to know that $H \wedge \overline{H}$ has this property — and it does.) □

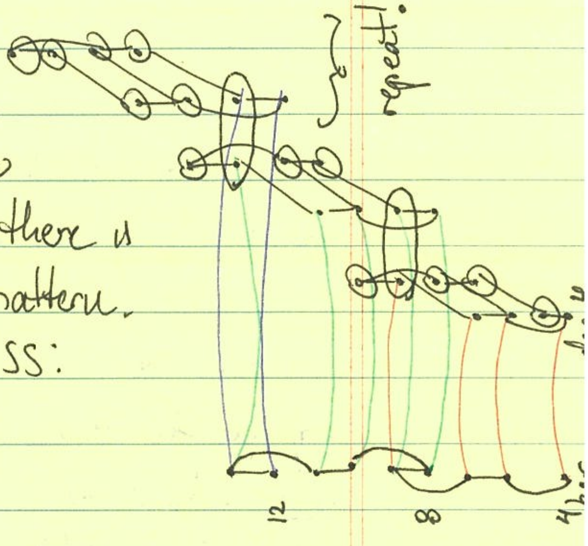
This construction is easier to think about after taking $\mathbb{F}_2$-duals. Then $E^1_{*,*} = (\pi_* H \wedge \overline{H}^{\wedge *} \wedge X)^* = \mathcal{A}^* \otimes \overline{\mathcal{A}}^{*\otimes *} \otimes H\mathbb{F}_2^* X$. The diff^l exactly tracks $\mathcal{A}^*$-module maps $H\mathbb{F}_2^* X \longrightarrow \mathbb{F}_2$, and a little homological algebra reveals

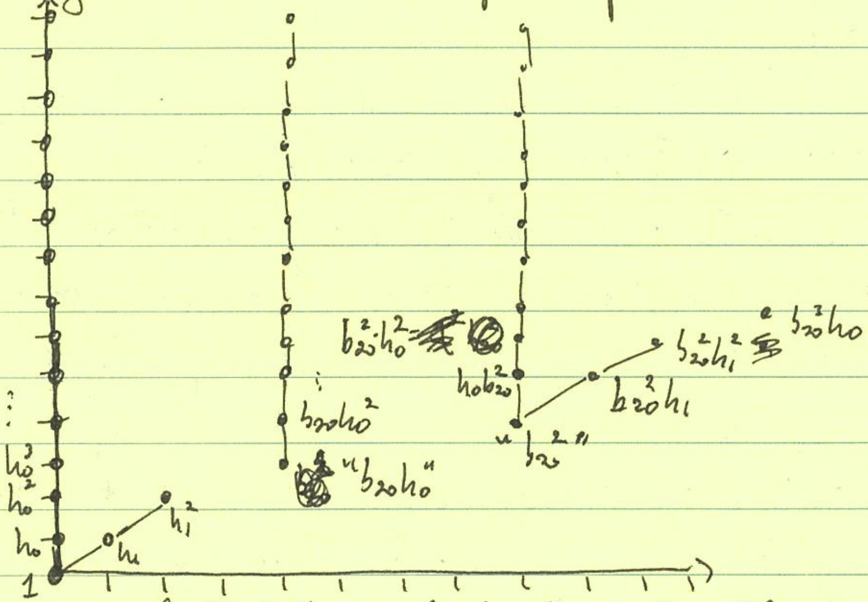
Lem: $E^2_{*,*} = \mathrm{Ext}^{\mathcal{A}^*\text{-mod}}_{*,*}(H\mathbb{F}_2^* X, \mathbb{F}_2)$. □

(Rem: Note that we're trying to compute $[\mathbb{S}^*, X]$, and our slogan ...)

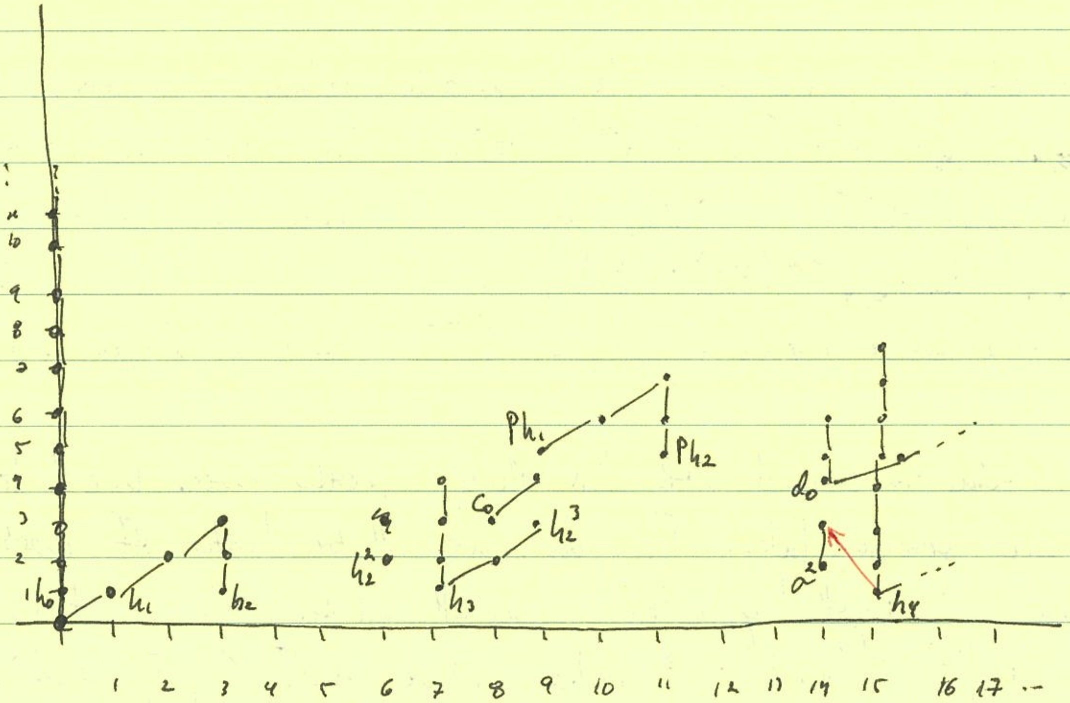
Actually learning to compute with this thing is pretty hard. We'll talk about it more seriously later. For now, we will be better off working a "toy example," so that you learn what this all feels like. It turns out that $\mathrm{Ext}^{\mathcal{A}(1)^*\text{-mod}}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_* kO$, where $\mathcal{A}(1)^* = \langle Sq^1, Sq^2 \rangle \subseteq \mathcal{A}^*$.

Pictured at right, we have manually computed a minimal free resolution of $\mathbb{F}_2$ in $A(1)^*$-modules. Its most interesting feature is that it repeats: the "bow shapes" beget more bow shapes, and each 4th step there is a lone dot that begets a whole new branch of the pattern. Hom-ing this into $\mathbb{F}_2$, we get a picture of our SS:





You can read off Bott periodicity from this. If only we could justify it...
Here's the version for $A^*$, the "full Adams SS":



# Hopf invariants and EHP fiber sequences.

Our goal today is to construct some important maps, called Hopf invariants, and to identify their fibers in one extremely important case.

The main ingredient is the following refl$^{\underline{n}}$ of algebra in topology:
Thm (James): For $X$ connected, $\Sigma(\Omega\Sigma X) \simeq \Sigma(\bigvee_{j=1}^{\infty} X^{\wedge j})$. □
The proof of this is actually surprisingly easy; the idea is to show it in mod-$p$ + $\mathbb{Q}$ homology $\forall p$, then use Hurewicz, and the whole business is modeled on the tensor algebra functor $T(M) = \bigoplus_j M^{\otimes j}$.

This splitting is highly nontrivial, and the inverse map that James's theorem guarantees is full of interesting information. Consider the following: the map $\Sigma\Omega\Sigma X \xrightarrow{\simeq} \Sigma(\bigvee_j X^{\wedge j}) \to \Sigma X^{\wedge 2}$ is adjoint to a map $\Omega\Sigma X \xrightarrow{h} \Omega\Sigma X^{\wedge 2}$, called the Hopf invariant. This construction is important, for instance, in the vector fields on spheres problem: an H-space structure on a sphere gives rise to a map

$$\begin{tikzcd}
CS^n \times S^n \arrow[rr] & & CS^n & & & \\
& S^n \times S^n \arrow[ul, hook] \arrow[dl, hook] \arrow[rr, "\mu"] & & S^n \arrow[ul, hook] \arrow[dl, hook] \arrow[r, squiggly] & S^n * S^n \arrow[r, "H\mu"] \arrow[d, phantom, "\cong"] & \Sigma S^n \arrow[d, phantom, "\cong"] \\
S^n \times CS^n \arrow[rr] & & CS^n & & S^{2n+1} \arrow[r, "H\mu"] & S^{n+1}
\end{tikzcd}$$

which interacts beautifully with h

For our purposes, we will want to set $X = S^n$ and to identify the homotopy fiber of $h$ (i.e., "$P_h$" from earlier in the course).
Recall: $H^*\Omega S^{2n+1} \cong \Gamma[x_{2n}]$ and $H^*\Omega S^{2n} \cong \Lambda[e_{2n-1}] \otimes \Gamma[x_{4n-2}]$.
We would like to analyze the fiber seq$^{\underline{ce}}$ $F \to \Omega S^{n+1} \xrightarrow{h} \Omega S^{2n+1}$ using the Serre spectral seq$^{\underline{ce}}$. We know the cohomology of the base and of the total space, and we know they are related by the edge homomorphism. If we can show the edge map is onto, then the spectral seq$^{\underline{ce}}$ will have to collapse. (Equivalently, we could show the map is surjective in homology.)

n odd: We have $H^* \underset{S^n}{\Omega} S^{2n+1} \longrightarrow H^* \underset{S^n}{\Omega} S^{n+1}$

$\Gamma[x_{2n}] \longrightarrow \Gamma[y_{2n}] \otimes \Lambda[e]$. If we can show $x_{2n} \longmapsto y_{2n}$, we will be done, using the algebra structure. This follows for formal / Freudenthal reasons: our map $h$ started life as the map $\Sigma \Omega \Sigma S^n \longrightarrow \Sigma (S^n)^{\wedge 2}$, which is an iso$^{\underline{m}}$ in cohomological degree $2n+1$. Tracing through the SSS then shows that $H^* F = \Sigma^{2n} \mathbb{Z}$, hence $F \simeq S^n$.

n even: This time we have $\Gamma[x_{2n}] \longrightarrow \Gamma[y_n]$ by $x_{2n} \longmapsto \frac{1}{2} y_n^2$. The algebra structure gives $x_{2n}^k \longmapsto \left(\frac{1}{2} y_n^2\right)^k = \frac{(2k)!}{2^k} \cdot y_n^{2k}$, which is a unit 2-locally. So, with $\mathbb{Z}_{(2)}$-coeff$^{s}$, we learn that $H^* F = \Sigma^n \mathbb{Z}_{(2)}$, so that $F \simeq S^n$ at the prime 2.

In fact, we can even identify the inclusion of $F$ as a familiar map. The Freudenthal map $S^n \xrightarrow{e} \Omega \Sigma S^n$ becomes null when postcomposed with $\Omega \Sigma S^n \xrightarrow{h} \Omega \Sigma (S^{2n})$ for connectivity reasons. However, we also know that the map $e$ is a cohomology iso$^{\underline{m}}$ in degree $n$ by Freudenthal, hence its factorization $S^n \xrightarrow{\simeq} F \longrightarrow \Omega \Sigma S^n$ is a homotopy equivalence.

Rem: In general, the identification of the fiber is harder because the cohomology of $X$ is not so sparse.

Rem: The map "e" is usually called this as an abbreviation of "Einhängung", German for "suspend". The continuation of this fiber seq$^{\underline{ce}}$ is

$$\cdots \longrightarrow \Omega S^n \xrightarrow{e} \Omega^2 S^{n+1} \xrightarrow{h} \Omega^2 S^{2n+1} \xrightarrow{p} S^n \xrightarrow{e} \Omega S^{n+1} \xrightarrow{h} \Omega S^{2n+1},$$

where "p" is short for "Whitehead product". The enterprising unstable homotopy theorist can read more in Neisendorfer's book.

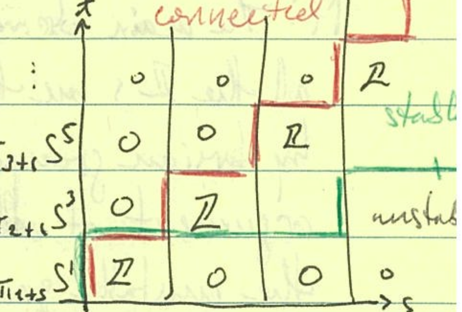
# Calculations in the EHPSS

~~[illegible]~~

The EHP fiber sequences knit together to give a homotopy SS:

$$\begin{tikzcd}
* \arrow[r] & \Omega^1 S^1 \arrow[r] \arrow[d, equal] & \Omega^2 S^2 \arrow[r] \arrow[d] & \Omega^3 S^3 \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^{n-1} S^{n-1} \arrow[r] \arrow[d] & \Omega^n S^n \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^\infty \mathbb{S}^0. \\
 & \Omega^1 S^1 & \Omega^2 S^3 & \Omega^3 S^5 & \cdots & \Omega^{n-1} S^{2n-3} & \Omega^n S^{2n-1} & \cdots,
\end{tikzcd}$$

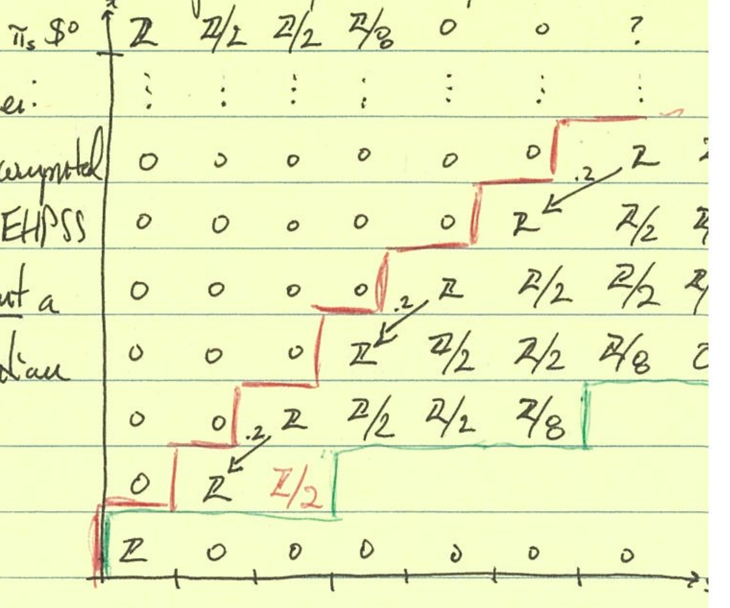
hence $E^1_{s,t} = \pi_s \Omega^t S^{2t-1} = \pi_{s+t} S^{2t-1} \Longrightarrow \pi_* \mathbb{S}^0$. This type sig. seems ridiculous: we start with all the unstable homotopy gp^s of spheres and manage to compute the stable ones... which we must have already known. The utility of this lies in the practical situation of <u>how</u> it does this computation, not the impossible situation of perfect information.



For all our technology, we can't fill out very much of this SS: we know a few stable groups and we know $\pi_{*\leq 6} S^3$. This is enough to make a small observation: the $s=1$ column is mostly empty, and it must receive a diff$^{\underline{l}}$ if it's to compute the expected $\mathbb{Z}/2 \overset{\circ}{\cong} \pi_1 \mathbb{S}^0$. [1]

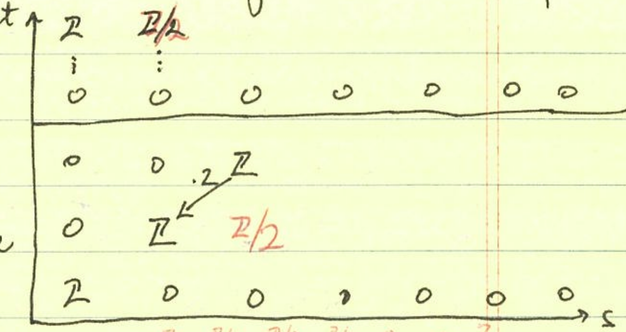
This is part of a family of diff$^{\underline{ls}}$, imported from the cellular structure of $\mathbb{RP}^\infty$ as in the diagram at right: the $\mathbb{Z}$'s on the main diagonal of the EHPSS participate in diff$^{\underline{ls}}$ given by mult. by $1+(-1)^s$.

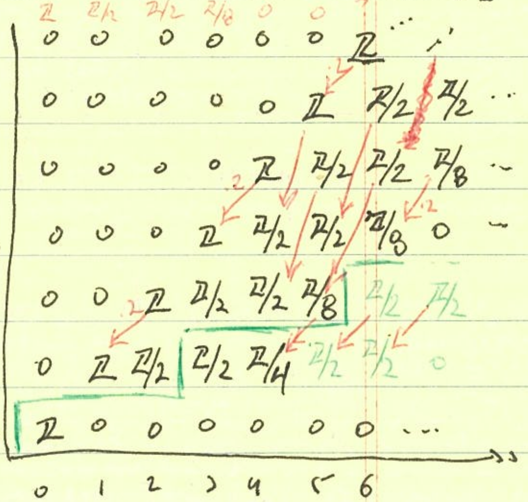
$$\begin{tikzcd}
\Omega^{n-1} S^{n-1} \arrow[r] & \Omega^n S^n \arrow[r] & \Omega^n S^{2n-1} \\
O(n-1) \arrow[r] \arrow[u] & O(n) \arrow[r] \arrow[u] & S^{n-1} \arrow[u] \\
\mathbb{RP}^{n-2} \arrow[r] \arrow[u] & \mathbb{RP}^{n-1} \arrow[r] \arrow[u] & S^{n-1} \arrow[u, equal]
\end{tikzcd}$$



The red group we know from many sources: $\pi_4 S^3$ lies in the stable range; we have computed $\pi_4 S^3$ unstably using Serre's method; if the EHPSS converges to $\mathbb{Z}/2$ in that column then <u>at least</u> a $\mathbb{Z}/2$ must be present; and a fourth truncation method which we now describe.

Instead of taking the colimit $\lim_{n\to\infty} \Omega^n S^n$, we can stop at any finite stage and compute $\pi_* \Omega^n S^n$ using a horizontal truncation of the full EHPSS, with the bottom $n$ rows surviving. For instance, here is a truncation converging to $\pi_* S^3$, so that its output loops back as input to the SS — of greater horizontal degree. This observation powers significant inductive computations.





Cor: $\pi_* S^{2n+1} \otimes \mathbb{Q} = \mathbb{Q}[2n+1]$

and $\pi_* S^{2n} \otimes \mathbb{Q} = \mathbb{Q}[2n] \oplus \mathbb{Q}[4n-1]$.

Pf: The main observation is that after the $E_2$-page, all the $\mathbb{Z}$'s on the main diagonal get replaced by torsion groups. The inductive/truncating argument at ~~odd~~ horizontal lines shows that the unstable groups $\pi_{*>2s+1} S^{2s+1}$ are all torsion too. However, truncating at an even horizontal line leaves one $\mathbb{Z}$ surviving (which "wants" to receive a $d_1$, but its source has been blanked out). That accounts for both parities. □

## The May spectral seq$^{\underline{ce}}$

Today we outline a more systematic approach to computing Adams $E^2$-terms, due to Peter May. The idea is that connected, graded, finite type Hopf algebras admit filtrations (essentially by "word length") which trivialize their multiplication and comultiplication. This gives a spectral seq$^{\underline{ce}}$ beginning w/ the cohomology of a bouquet of exterior algebras.

Thm: (May) There is a spectral seq$^{\underline{ce}}$ of algebras with $E_1^{*,*,*} = \mathbb{F}_2[h_{ij} \mid i \geq 1, j \geq 0]$, converging to the Adams $E_2$-term for the sphere. □

Lem: $h_{ij}$ represents $\xi_i^{2^j}$, hence $d_1 h_{ij} = \sum_{k=1}^{i-1} h_{kj} h_{(i-k)(k+j)}$. □

Lem: There is a version of this spectral seq$^{\underline{ce}}$ converging to the Adams $E_2$-page for $\pi_* \hat{ko}_2$, beginning with just $\mathbb{F}_2[h_{10}, h_{11}, h_{20}]$. □

Will talk about this smaller example in a language that generalizes. The specification of the seq above is multiplicative, in the sense that the Leibniz rule specifies everything. Computations can be made linear again by working with $E_1^2 = \mathbb{F}_2[h_{10}^2, h_{11}^2, h_{20}^2]$-modules.

$$\begin{array}{r|cccccccc}
E_1 = E_1^2\{ & 1, & h_{10}, & h_{11}, & h_{20}, & h_{10}h_{11}, & h_{11}h_{20}, & h_{10}h_{20}, & h_{10}h_{11}h_{20}\} \\
\hline
E_1 = E_1^2\{1, & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{10}^2 h_{11}^2 \\
h_{10} & 0 & 0 & 0 & 0 & 0 & h_{11}^2 & 0 & 0 \\
h_{11} & 0 & 0 & 0 & 0 & 0 & 0 & h_{10}^2 & 0 \\
h_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h_{10}h_{11} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
h_{11}h_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h_{10}h_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h_{10}h_{11}h_{20}\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

$d_1$

$\Longrightarrow Z = E_1^2\{1, h_{10}, h_{11}; h_{10}h_{11}\}$

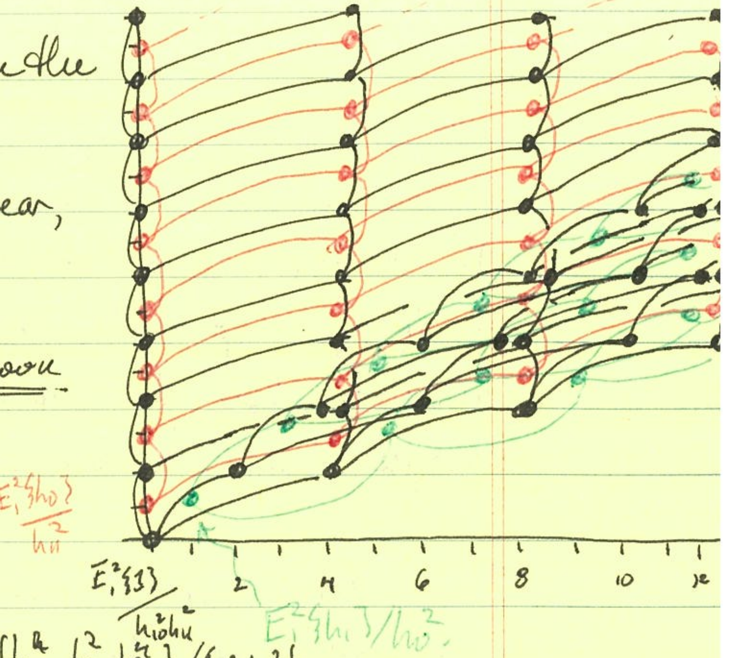
$\Longrightarrow B = \langle 1 \cdot h_{10}h_{11}, h_{11}^2 \cdot h_{10}, h_{10}^2 \cdot h_{11}, h_{10}^2 h_{11}^2 \cdot 1 \rangle$

$\Downarrow$

$$E_2 = H = E_1^2 \left\{ \frac{1}{h_{10}^2 h_{11}^2}, \frac{h_{10}}{h_{11}^2}, \frac{h_{11}}{h_{10}^2} \right\}.$$

To continue the calculation, we need some way to describe longer differentials in the spectral seq$^{\underline{ce}}$.

Thm (Nakamura): There are operators $Sq^n$ acting on the May spectral seq$^{ce}$ satisfying...
(i) $Sq^0 h_{ij} = h_{i(j+1)}$, (ii) $Sq^1 h_{ij} = (h_{ij})^2$, (iii) $Sq^n$ is linear,
(iv) $Sq^n(x \cdot y) = \Sigma_{i+j=n} Sq^i(x) \cdot Sq^j(y)$, and
(v) $Sq^n(d_? x) = d_? Sq^n(x)$, where "?" is unknown



Ex: $d_?(h_{20}^2) = d_?(Sq^1 h_{20}) = Sq^1 d_1 h_{20}$

$= Sq^1(h_{10}h_{11}) = Sq^1 h_{10} Sq^0 h_{11} + Sq^0 h_{10} Sq^1 h_{11} = h_{10}^2 h_{12} + h_{11}^3$.

We compute the subalgebra of squares to be $E_2^2 = \mathbb{F}_2[h_{10}^4, h_{11}^2, h_{20}^4]/(h_{10}^2 h_{11}^2)$.
and continuing in this way gives a computation of $d_2$ as an $E_2^2$-linear map.

This has kernel $Z = \langle 1, h_{10}, h_{11}, h_{10}h_{20}^2, h_{10}^2 \cdot h_{20}^2 \rangle$ and image $B = \langle h_{11}^4 \cdot 1, h_{11}^2 \cdot h_{11} \rangle$, hence cohomology

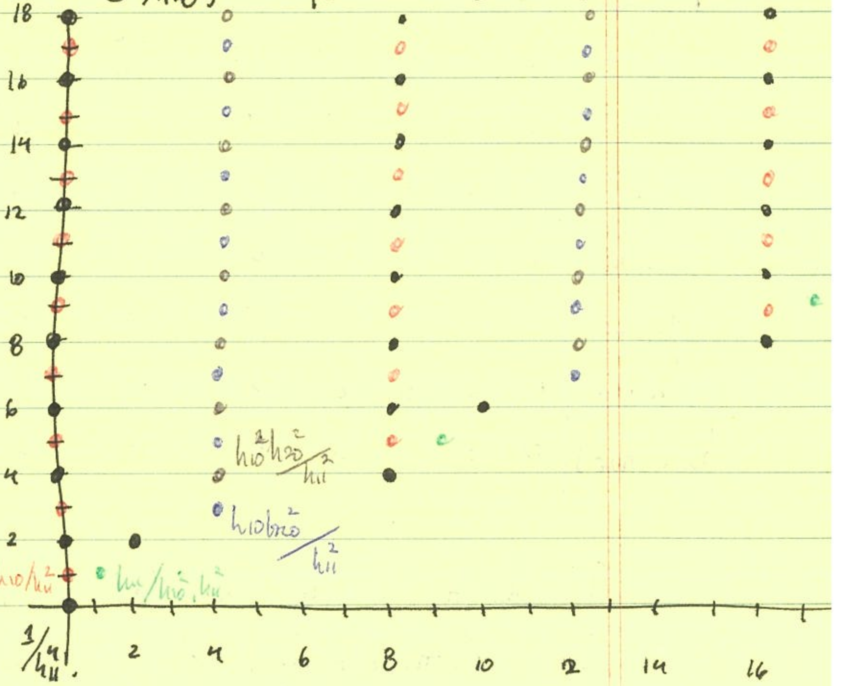
$$E_3 = H = E_2^2 \frac{\{1\}}{h_{11}^4} \oplus \frac{\{h_{10}\}}{h_{11}^2} \oplus \frac{\{h_{11}\}}{(h_{10}^2, h_{11}^2)} \oplus \frac{\{h_{10}h_{20}^2\}}{h_{11}^2} \oplus \frac{\{h_{10}^2 h_{20}^2\}}{h_{11}^2}.$$

$$d_2\quad E_2^* = E_2^2\{1\} \oplus \frac{\{h_{10}\}}{h_{11}^2} \oplus \frac{\{h_{11}\}}{h_{10}^2} \oplus \frac{\{h_{10}h_{20}^2\}}{h_{11}^2} \oplus \frac{\{h_{11}h_{20}^2\}}{h_{10}^2} \oplus$$

| | | $h_{11}^2$ | $h_{10}^2$ | $h_{11}^2$ | $h_{10}^2$ | |
|---|---|---|---|---|---|---|
| $E_2 = E_2^2\{1\}$ | 0 | 0 | 0 | 0 | $h_{11}^4$ | 0 |
| $\oplus\{h_{10}\}/h_{11}^2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\oplus\{h_{11}\}/h_{10}^2$ | 0 | 0 | 0 | 0 | 0 | $h_{11}^2$ |
| $\oplus\{h_{10}h_{20}^2\}/h_{11}^2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\oplus\{h_{11}h_{20}^2\}/h_{10}^2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\oplus\{h_{20}^2\}$ | 0 | 0 | 0 | 0 | 0 | 0 |

This is the last page; we include a sketch below:

The general computation is similar in techniques, but vastly more complicated. (For $A(2)$, it still turns out to be completely feasible, and was first worked out (I think...!) by Mahowald + Hopkins).



## Where to go from here (Oh, the places you'll go!)

You now know enough algebraic topology to be dangerous. (Anyone who hears "spectrum" and doesn't flinch is dangerous.) However, we're still a ways from the forefront of the field, and I wanted to point out some current pedagogical landmarks, c. 2017 (+ tailored to my preferences, of course).

- A lot of what we learned in this class feeds directly into the "*vector fields on spheres*" problem, a major accomplishment of 1970's topology, and among the first conceptual problems solved by broad-reaching computational invention. This is totally a "next step" for this class.
- Stable homotopy houses a lot of geometry through "*bordism homology*", where formal sums of singular simplices are replaced by maps in from manifolds, w/ $\partial$. Even the bordism homology of a point is of great interest — it houses a kind of homotopical intersection theory for manifolds.
- *Algebraic K-theory* is a generic tool that captures a lot of geometric information about any kind of context: alg. geom., manifolds, groups, .... It's kind of bottomless, and being able to compute anything about it is often met with cheers + wild success.
- *Homological stability* refers to the phenomenon that "things", like symmetry groups, often occur in families, like $\{\Sigma_m\}_{m=1}^{\infty}$, and that these families have highly compatible homologies. This comes in many shapes and sizes + generally has geometric content.
- *Equivariant homotopy theory* is of geometric interest because symm. gp^s appear everywhere in geometric contexts, and requiring homotopy theory to be mindful of it makes homotopy theory itself considerably more geometric.

- Homotopy theory has surprising applications in algebraic geometry. Artin–Mazur gives access to a homotopy type associated to a site on a scheme, and e.g. the étale homotopy type carries a remarkable amount of information about the scheme. Meanwhile, motivic homotopy theory is a modern blend of homotopical techniques with algebro-geometric ones, with large implications for both sides: Voevodsky used this to settle the Milnor/Bloch–Kato conjectures, and Isaksen has used this to push $\pi_* \mathbb{S}$ computations considerably further.
- Goodwillie calculus describes a natural seq^ce of stable invariants to almost any homotopical functor, and it turns out that these invariants generalize/unify a bunch of unrelated classical invariants from practically every corner of homotopy theory.
- Spectral algebra(ic geometry) is a re-encoding of classical algebra into spectra. We talked about "ring spectra" in this class, but it turns out that asking for, e.g., associativity requires real heavy machinery. The rewards are great: the theory of "modules" associated to such a spectrum is very rich + eases a lot of arguments, and the extra operations impressed on such a spectrum encode useful (and often classically relevant) data. (One can then try "doing algebraic geometry" in this setting, which appears to be a hole w/o bottom.)
- There is an older program in homotopy theory, called chromatic homotopy theory, which continues to give the tightest results on the "global" behavior of homotopy theory by comparing it to particular (highly complex) algebraic models. You can get a lot of intuition very quickly by learning some of this. Its genesis was in understanding certain periodic phenomena in the Adams spectral seq^ce — hence is often taught from a highly computational point of view, if that's your thing.