AN INTRODUCTION TO MODEL CATEGORIES

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1. Why model categories?

1.1. **Localizing categories.** Let $\mathbf{C}$ be a category and $\mathbf{W} \subseteq \mathrm{Mor}(\mathbf{C})$ a collection of morphisms, which we think of as (weak) equivalences. Often, one wants to treat the morphisms in $\mathbf{W}$ as isomorphisms. An example is the homotopy category of spaces $\mathrm{Ho}(\mathbf{Spaces}_{*/})$: we have $\mathbf{C} = \mathbf{Spaces}_{*/}$ and $\mathbf{W}$ the collection of homotopy equivalences.

There is a general way, called *Gabriel-Zisman localization*, to form a new category $\mathbf{C}[\mathbf{W}^{-1}]$ where the equivalences are "formally inverted". Roughly speaking, $\mathbf{C}[\mathbf{W}^{-1}]$ has

- objects the same as $\mathbf{C}$, and
- a morphism $x \to y$ in $\mathbf{C}[\mathbf{W}^{-1}]$ is a zig-zag of arrows
$$x \to \bullet \xleftarrow{\sim} \bullet \to \cdots \to \bullet \xleftarrow{\sim} \bullet \to y$$
in $\mathbf{C}$, where the backward arrows belong to $\mathbf{W}$, modulo the obvious relations.

However, there is a problem with this construction. Namely, it may be "too large" to even be a category (at least not a locally small one). Even if you sweep these set-theoretic issues under the rug, there is still the problem of not having a good handle on the morphisms in $\mathbf{C}[\mathbf{W}^{-1}]$. The situation is slightly better if we have a calculus of fractions.

The theory of model categories furnishes another possible solution to this problem. By equipping the category with weak equivalences $(\mathbf{C}, \mathbf{W})$ with additional structure, one can construct its homotopy category, which I'll eventually denote $\mathrm{Ho}(\mathbf{C})$, which can be shown to be a localization of $\mathbf{C}$ with respect to $\mathbf{W}$, i.e., $\mathrm{Ho}(\mathbf{C})$ and $\mathbf{C}[\mathbf{W}^{-1}]$ are equivalent as categories. Furthermore, there is a more manageable description of the morphisms in $\mathrm{Ho}(\mathbf{C})$, amenable to calculation.

But it's important to keep in mind that what's actually important is the underlying category with weak equivalences $(\mathbf{C}, \mathbf{W})$ – the other structures ought to be dispensable. One way to see this is to note that there is a less naïve localization, called *hammock localization*[1] ([DK80a], [DK80b]), intermediate between the model category $\mathbf{C}$ and its homotopy category $\mathrm{Ho}(\mathbf{C})$. This is a universal simplicial category $L^H\mathbf{C}$ in which morphisms in $\mathbf{W}$ are invertible. In particular, regarding $\mathbf{C}[\mathbf{W}^{-1}]$ as a discrete simplicial category, we have the picture
$$\mathbf{C} \to L^H\mathbf{C} \to \mathbf{C}[\mathbf{W}^{-1}].$$

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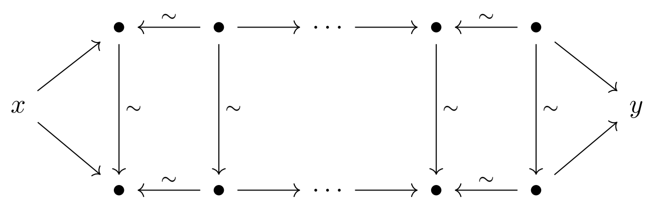
[1]Possible paper topic #11.

The map on mapping spaces in $L^H\mathbf{C} \to \mathbf{C}[\mathbf{W}^{-1}]$ is given by projection to path components. Philosophically, the problem with $\mathbf{C}[\mathbf{W}^{-1}]$ is that it has forgotten too much structure. Another lesson we may draw from this is that localization of categories is inherently homotopical.

So what is this hammock localization? $L^H\mathbf{C}$ is the simplicially enriched category whose

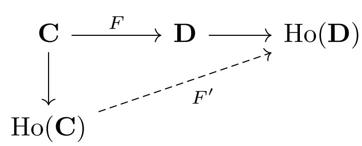
- objects are the same as $\mathbf{C}$
- the 0-simplices in $L^H\mathbf{C}(x, y)$ are just the zigzags from above,
- the 1-simplices are "hammocks":



- the n-simplices are "wider hammocks" of width n. In other words, $L^H\mathbf{C}(x, y)$ is the nerve of the category of zigzags from x to y with weak equivalences of zigzags, modulo some relations.

Hammock localization behaves well with model categories. For instance, let $\mathbf{C}$ be a simplicial model category, and $\mathbf{C}_0$ its underlying category. Then $\mathbf{C} \simeq L^H(\mathbf{C}_0)$ as simplicial categories.

1.2. **Derived functors.** Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between categories with weak equivalences. A (total) *derived functor* F' of F is a universal functor in the diagram



If F is *homotopical* – it takes weak equivalences in $\mathbf{C}$ to weak equivalences in $\mathbf{D}$ – then F' would be easy to define, but most functors are not homotopical. The derived functor F' is then a best homotopical approximation to F.

In general, when do derived functors exist and how are they defined? Again, category theory has an all-purpose solution to these kinds of questions: *Kan extensions*. However, there are problems with this naïve answer: functors produced in this fashion are not typically pointwise Kan extensions since $\mathrm{Ho}(\mathbf{D})$ may lack limits and colimits, so for example the composite of two derived functors is not necessarily the derived functor of the composite.

Model categories allows one to restrict attention to nicely behaved functors that do not suffer from such defects.

Remark 1.1. Again, a full model structure is not required for the construction of derived functors. In [DHKS04], the authors develop the notion of a *homotopical category* and *deformable functors*, which is sufficient. Another alternative is to work in the framework of the Dwyer-Kan simplicial localization.

1.3. **Abstract homotopy theory.** By "homotopy theory", one usually means the homotopy theory of topological spaces. We study homotopy equivalences, fibrations, and cofibrations of spaces and how they interact in homotopically meaningful ways. In the 1960s, Quillen observed that one can do "homotopy theory" in other settings. In fact, the theory of model categories is modelled heavily on the theory of homological algebra, where one does "homotopy theory" with chain complexes up to quasi-isomorphisms. Quillen also wrote down the main features of this "homotopical algebra" – a theory of homotopy theory, if you will. Constructions and theorems that can be proven in this framework will apply to all these diverse situations.

Our goal in this note will be modest[2]. We'll define what a model category is, carefully construct the homotopy category $\mathrm{Ho}(\mathbf{C})$ of a model category $\mathbf{C}$, and discuss derived functors and hopefully also homotopy limits and colimits.

[2] In retrospect, not so much.

1.4. $(\infty, 1)$**-categories.** Granting that model categories is supposed to provide a "theory of homotopy theories", what about a "*homotopy* theory of homotopy theories"? One way to formulate this question is to ask whether there is a model category of model categories in which the weak equivalences are, say, the Quillen equivalences (defined later). As far as I know, this has not been fully worked out. Instead, one solution is to use the theory of $(\infty, 1)$-*categories*[3], aka ∞-*categories* or *quasicategories*, developed by Joyal and Lurie. There is an ∞-category of ∞-categories, which is nice for example if we want to take (homotopy) limits of categories.

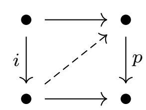
Many ∞-categories come to us from model categories. For example, the *coherent nerve* of a locally fibrant simplicial model category (which one can obtain via simplicial localization and fibrant replacement) is a ∞-category. Call an ∞-category *presentable* if it is cocomplete and accessible. Then an ∞-category is presentable if and only if it is equivalent to the underlying ∞-category of some combinatorial simplicial model category. In layman's terms, model categories gives us a way to *present* ∞-categories, analogous to how generators and relations are used to present groups.

For more information, see [Lur09].

1.5. **References.** The original source for this material is Quillen's *Homotopical algebra* [Qui67], which is still useful to read not only for cultural edification. I will closely follow the excellent introductory article [DS95]. While preparing this, I also benefited from reading some of Emily Riehl's writings, e.g., [Rie14]. Other more comprehensive treatments of model categories can be found in [Hir03] (read part two first) and [Hov99].

2. Definitions, examples, and basic properties

Definition 2.1 (Lifting properties)**.** Consider a diagram of the shape



If the dotted arrow exists (for arbitrary horizontal arrows), we say that i has the *left lifting property with respect to* p. Dually, we also say that p has the *right lifting property with respect to* i.

Definition 2.2. A *(closed) model category* is a category $\mathbf{C}$ with three distinguished wide subcategories:

(i) *weak equivalences* $\mathbf{W}$, whose morphisms will be denoted ($\xrightarrow{\sim}$);
(ii) *fibrations* **fib**, whose morphisms will be denoted ($\twoheadrightarrow$); and
(iii) *cofibrations* **cof**, whose morphisms with be denoted ($\rightarrowtail$);

satisfying the following axioms:

(MC1) $\mathbf{C}$ is bicomplete.
(MC2) Weak equivalences satisfy the 2-out-of-3 property.
(MC3) Weak equivalences, fibrations, and cofibrations are closed under retracts.
(MC4)
- Cofibrations have the left lifting property with respect to acyclic fibrations.
- Fibrations have the right lifting property with respect to acyclic cofibrations.

(MC5) Any map f can be functorially factored in two ways:
- as a cofibration followed by an acyclic fibration, or
- as an acyclic cofibration followed by a fibration.

The last two axioms say that $(\mathbf{cof}, \mathbf{W} \cap \mathbf{fib})$ and $(\mathbf{W} \cap \mathbf{cof}, \mathbf{fib})$ form *weak factorization systems*.

Remark 2.3. These axioms can be weakened. For example, Quillen's original definition only asked for finite limits and colimits, and factorizations need not be functorial. However, we'll often want more limits and colimits for Quillen's small object argument, which produces functorial factorizations automatically anyway.

Remark 2.4. The definition of a model category is dual: if $(\mathbf{C}, \mathbf{W}, \mathbf{fib}, \mathbf{cof})$ is a model category, then so is $(\mathbf{C}^{\mathrm{op}}, \mathbf{W}^{\mathrm{op}}, \mathbf{cof}^{\mathrm{op}}, \mathbf{fib}^{\mathrm{op}})$. Upshot: every time we prove a statement we automatically get a dual statement for free.

[3] Possible paper topic # 10.

Definition 2.5. A model category $\mathbf{C}$ has an initial object $\emptyset$ and a terminal object $*$. An object A in $\mathbf{C}$ is

(i) *cofibrant*, if the map $\emptyset \to A$ is a cofibration; and
(ii) *fibrant*, if the map $A \to *$ is a fibration.

Examples 2.6.

(a) The category **Spaces** of topological spaces is a model category where a map is
   (i) a weak equivalence if it is a weak homotopy equivalence,
   (ii) a fibration if it is a Serre fibration[4],
   (iii) a cofibration if it is a retract of a cell complex, i.e., a map obtained by successively attaching cells.

   Every object is fibrant, and the cofibrant objects are retracts of "generalized CW complexes".

(b) There is another model structure on **Spaces** where the weak equivalences are homotopy equivalences (recall for maps between non-CW complexes, Whitehead's theorem need not hold). The fibrations are *Hurewicz fibrations*, and the cofibrations are *closed Hurewicz cofibrations*, i.e., maps $A \to B$ with the homotopy extension property in which $A \subseteq B$ is a closed subspace. See [Str72].

(c) The category $\mathbf{Ch}_R$ of nonnegatively graded chain complexes over R is a model category where a chain map is
   (i) a weak equivalence if it is a quasi-isomorphism,
   (ii) a cofibration if it is a degree-wise monomorphism with projective cokernel in all nonnegative degrees, and
   (iii) a fibration if it is a degree-wise epimorphism in positive degrees.

   This is called the *projective model structure.*

   There are several other model structures one may put on $\mathbf{Ch}_R$. For example, call a chain map
   (i) a weak equivalence if it is a quasi-isomorphism,
   (ii) a cofibration if it is a degree-wise monomorphism in positive degrees, and
   (iii) a fibration if it is a degree-wise epimorphism with injective kernel in all nonnegative degrees.

   This forms the *injective model structure* on $\mathbf{Ch}_R$.

(d) Simplicial sets[5] also form a model category. Call a map f
   (i) a weak equivalence if it is a weak homotopy equivalence after geometric realization,
   (ii) a cofibration is it is a levelwise injection, and
   (iii) a fibration if it is a *Kan fibration*, i.e., maps with the rlp against all horn inclusions.

   In this *Quillen/Kan model structure*, every simplicial set is cofibrant and the fibrant objects are Kan complexes.

   Alternatively, call f
   (i) a weak equivalence if it is a *weak categorical equivalence*,
   (ii) a cofibration if it is a levelwise injection, and
   (iii) a fibration if it is an "inner Kan fibration", i.e., maps with the rlp against inner horn inclusions.

   In this *Joyal model structure*, every object is cofibrant and the fibrant objects are ∞-categories.

(e) Simplicial objects in abelian categories. See [Qui67].

Exercise 2.7. Check that these do form model categories. Some of the axioms should be familiar, while others require some work.[6]

Let $I \subset \mathrm{Mor}(\mathbf{C})$ and $f \in \mathrm{Mor}(\mathbf{C})$. We write

- $f \in \mathrm{llp}(I)$ if f has the left lifting property with respect to all morphisms in I, and
- dually, $f \in \mathrm{rlp}(I)$ if f has the right lifting property with respect to all morphisms in I.

Proposition 2.8. *Let* $\mathbf{C}$ *be a model category. Then*

(i) $\mathbf{cof} = \mathrm{llp}(\mathbf{W} \cap \mathbf{fib})$.
(ii) $\mathbf{W} \cap \mathbf{cof} = \mathrm{llp}(\mathbf{fib})$.
(iii) $\mathbf{fib} = \mathrm{rlp}(\mathbf{W} \cap \mathbf{cof})$.
(iv) $\mathbf{W} \cap \mathbf{fib} = \mathrm{rlp}(\mathbf{cof})$.

[4]Eric calls these "weak fibrations" to distinguish them from what I call Hurewicz fibrations below.
[5]Possible paper topic # 14.
[6]See some of the problems on the problem set.

Proof of (i). We'll give a proof of (i); (ii) is similar and the others follow from duality.

Given $f : X \in Y$ with the llp against acyclic fibrations, factor it as $f : X \overset{i}{\rightarrowtail} Y' \overset{\sim}{\twoheadrightarrow} Y$. There is a lift $Y \to Y'$ in the diagram

$$\begin{array}{ccc} X & \overset{i}{\rightarrowtail} & Y' \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow\!\!\!\downarrow{\scriptstyle \sim} \\ Y & = & Y, \end{array}$$

which realizes f as a retract of the cofibration i. So f is a cofibration. □

This means that given the weak equivalences, cofibrations determine the fibrations and vice-versa.

Proposition 2.9. *Let* $\mathbf{C}$ *be a model category. Then* $\mathbf{cof}$ *and* $\mathbf{W} \cap \mathbf{cof}$ *are stable under cobase change, and* $\mathbf{fib}$ *and* $\mathbf{W} \cap \mathbf{fib}$ *are stable under base change.*

Proof. Easy exercise using the previous proposition. □

3. Homotopy relations

Fix a model category $\mathbf{C}$.

Definition 3.1. Let A be an object in $\mathbf{C}$. A *(good) cylinder object* for A is an object $\mathrm{cyl}(A)$ in $\mathbf{C}$ together with a factorization

$$A \sqcup A \overset{i}{\rightarrowtail} \mathrm{cyl}(A) \overset{\sim}{\longrightarrow} A$$

of the fold map $\nabla : A \sqcup A \to A$.

Two maps $f, g : A \to X$ in $\mathbf{C}$ are said to be *left homotopic*, written $f \overset{l}{\sim} g$, if there exists a cylinder object $\mathrm{cyl}(A)$ for A so that the sum map $f \sqcup g : A \sqcup A \to X$ extends to a map $H : \mathrm{cyl}(A) \to X$.

Remark 3.2. We are adopting Hovey's requirement that a cylinder object be good. In [DS95], the map $i : A \sqcup A \to \mathrm{cyl}(A)$ is not assumed to be a cofibration.

Example 3.3. In **Spaces**, $A \times I$ is a cylinder object for A if A is cofibrant, hence the name. (The assumption that A is cofibrant is so that $A \sqcup A = A \times \partial I \hookrightarrow A \times I$ is a cofibration, otherwise the cylinder object isn't good.)

Lemma 3.4. *If* A *is cofibrant, the maps* $i_0, i_1 : A \to \mathrm{cyl}(A)$ *are acyclic cofibrations.*

Proof. Since composing with $\mathrm{cyl}(A) \overset{\sim}{\longrightarrow} A$ gives the identity which is a weak equivalence, the 2-of-3 property implies i_0 and i_1 are weak equivalences. Now, if A is cofibrant, we obtain in_0 and in_1 as pushouts of a cofibration

$$\begin{array}{ccc} \emptyset & \rightarrowtail & A \\ \downarrow & & \downarrow{\scriptstyle \mathrm{in}_0} \\ A & \underset{\mathrm{in}_1}{\longrightarrow} & A \sqcup A \end{array}$$

so they're cofibrations too. Now, i_0 is the composition of two cofibrations

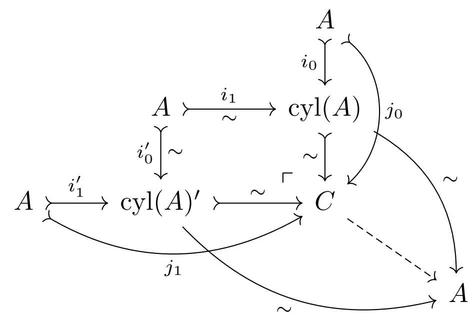
$$i_0 : A \overset{\mathrm{in}_0}{\rightarrowtail} A \sqcup A \rightarrowtail \mathrm{cyl}(A)$$

so it's a cofibration too. Similar for i_1. □

Proposition 3.5. *If* A *is cofibrant, then* $\overset{l}{\sim}$ *is an equivalence relation on* $\mathbf{C}(A, X)$.

Proof. Reflexivity and symmetry are obvious. For transitivity, suppose $f \overset{l}{\sim} g$ and $g \overset{l}{\sim} h$, and choose good left homotopies $H : \mathrm{cyl}(A) \to X$ from f to g and $H' : \mathrm{cyl}(A)' \to X$ from g to h.

We'll define a new object C by pushing out:



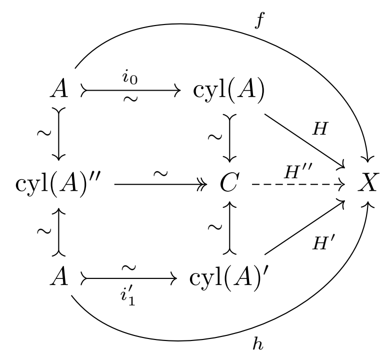
Since A is cofibrant, i_0, etc. are all acyclic cofibrations. By stability under cobase change, $\mathrm{cyl}(A) \to C$ are acyclic cofibrations. Since additionally $\mathrm{cyl}(A) \to A$ is a weak equivalence, we deduce that $C \to A$ is also a weak equivalence by the 2-of-3 property.

It might bother you that the map $j_0 \sqcup j_1 : A \sqcup A \to C$ is not a cofibration. This is not an issue since we can just factor this as a cofibration followed by a acyclic fibration:

$$A \sqcup A \rightarrowtail \mathrm{cyl}(A)'' \overset{\sim}{\twoheadrightarrow} C.$$

(This is why we can concern ourselves with just good cylinder objects, rather than all cylinder objects.)

Finally, by the universal property of a pushout there is a map $H'' : C \to X$, which is a left homotopy from f to h.



$\square$

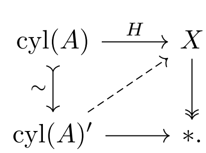
Let $\pi^l(A, X)$ denote the set of equivalence classes of $\mathbf{C}(A, X)$ under the equivalence relation generated by left homotopy.

Lemma 3.6. *If X is fibrant, $f \overset{l}{\sim} g : A \to X$, and $h : A' \to A$, then $fh \overset{l}{\sim} gh$.*

Proof. Let $H : \mathrm{cyl}(A) \to X$ be a left homotopy from f to g. We can assume the structure map $\mathrm{cyl}(A) \xrightarrow{\sim} A$ is not only a weak equivalence but an acyclic fibration. (This is what it means to be a *very good* cylinder object in [DS95].) To see this, factor the map into an acylic cofibration followed by a (necessarily acyclic, by 2-of-3) fibration:

$$\mathrm{cyl}(A) \overset{\sim}{\rightarrowtail} \mathrm{cyl}(A)' \overset{\sim}{\twoheadrightarrow} A.$$

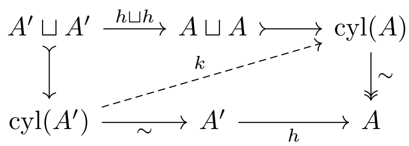
Then since X is fibrant, we can extend H over $\mathrm{cyl}(A)'$:



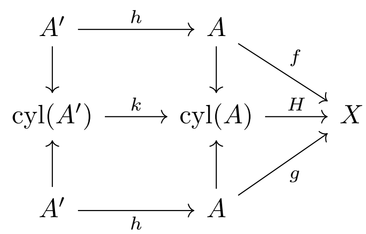
Now let

$$A' \sqcup A' \rightarrowtail \mathrm{cyl}(A') \xrightarrow{\sim} A'$$

be a cylinder object for A'. Construct a lift of the following diagram



Then Hk is a left homotopy from fh to gh:



□

Corollary 3.7. *If X is fibrant, then the composition in* **C** *induces a well-defined map*

$$\pi^l(A', A) \times \pi^l(A, X) \to \pi^l(A', X).$$

Proof. We need to show that if $h \overset{l}{\sim} k : A' \to A$ and $f \overset{l}{\sim} g : A \to X$, then fh and gk represent the same element of $\pi^l(A', X)$. We showed in the previous lemma that $fh \overset{l}{\sim} gh$. So it remains to show that $gh \overset{l}{\sim} gk$. This is easier: just compose the homotopy between h and k with g. □

By duality, one can formulate the notion of *path objects* factoring the diagonal map and right homotopies and prove dual lemmas.

Definition 3.8. A *(good) path object* for X is an object $\mathrm{cocyl}(X)$ with a factorization

$$X \xrightarrow{\sim} \mathrm{cocyl}(X) \overset{p}{\twoheadrightarrow} X \times X$$

of the diagonal map $\Delta : X \to X \times X$.

Two maps $f, g : A \to X$ are *right homotopic*, written $f \overset{r}{\to} g$ if there is a path object $\mathrm{cocyl}(X)$ such that the product map $f \times g : A \to X \times X$ lifts to a map $H : A \to \mathrm{cocyl}(X)$.

Example 3.9. The path space X^I is a path object for X in **Spaces**.

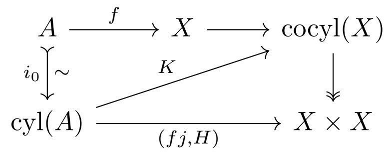
The relationship between left and right homotopy is given by:

Lemma 3.10. *Let $f, g : A \to X$ be maps.*

(a) If A is cofibrant and $f \overset{l}{\sim} g$, then $f \overset{r}{\sim} g$.

(b) If X is fibrant and $f \overset{r}{\sim} g$, then $f \overset{l}{\sim} g$.

Proof of (a). Let $H : \mathrm{cyl}(A) \to X$ be a left homotopy from f to g. Denote by j the structure map $\mathrm{cyl}(A) \to A$. Since A is cofibrant, we've seen that $i_0 : A \to \mathrm{cyl}(A)$ is an acyclic cofibration. Put this together in the diagram



Then $Ki_1 : A \to \mathrm{cocyl}(X)$ is the desired right homotopy. □

If A is cofibrant and X is fibrant, we'll denote the coincident equivalence relation by $\sim$ and say that two equivalent maps are *homotopic*. The set of equivalence classes is denoted by $\pi(A, X)$.

We end this section with an application from topology.

Theorem 3.11 (Whitehead)**.** *Let $f : A \to X$ be a map between bifibrant objects. Then f is a weak equivalence iff it is a homotopy equivalence.*

Lemma 3.12. *If A is cofibrant and $p : Y \to X$ is an acyclic fibration, then composition with p induces a bijection $p_* : \pi^l(A, Y) \to \pi^l(A, X)$, $[f] \mapsto [pf]$.*

Proof. The map p_* is well-defined: if H is a left homotopy from f to g, then pH is a left homotopy from pf to pg.

Surjectivity: let $[f] \in \pi^l(A, X)$. Find a lift $g : A \to Y$ in

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \big\downarrow & \overset{g}{\nearrow} & \sim \big\downarrow p \\ A & \xrightarrow[f]{} & X \end{array}$$

Then $p_*[g] = [pg] = [f]$.

Injectivity: let $f, g : A \to Y$ and suppose $pf \overset{l}{\sim} pg : A \to X$. Consider a left homotopy $H : \mathrm{cyl}(A) \to X$ from pf to pg. Find a lift $\tilde{H}$ in

$$\begin{array}{ccc} A \sqcup A & \xrightarrow{f \sqcup g} & Y \\ \big\downarrow & \overset{\tilde{H}}{\nearrow} & \sim \big\downarrow p \\ \mathrm{cyl}(A) & \xrightarrow[H]{} & X. \end{array}$$

Then $\tilde{H}$ is a homotopy from f to g. ◻

Proof of theorem 3.11. Suppose $f : A \xrightarrow{\sim} X$ is a weak equivalence. Factor it as an acyclic cofibration followed by an acyclic fibration:

$$A \underset{\sim}{\overset{q}{\rightarrowtail}} C \underset{\sim}{\overset{p}{\twoheadrightarrow}} X.$$

Observe that C is bifibrant. Construct a lifting s in the diagram

$$\begin{array}{ccc} \emptyset & \rightarrowtail & C \\ \big\downarrow & \overset{s}{\nearrow} & \sim \big\downarrow p \\ X & = & X \end{array}$$

Then $ps = \mathrm{id}_X$. We claim that s is a two-sided homotopy inverse to p. Consider the homotopy class $[sp]$. Then $p_*[sp] = [psp] = [p] = p_*[\mathrm{id}_C]$. So by the previous lemma, $[sp] = [\mathrm{id}_C]$ as wanted.

Dually, there exists a two-sided homotopy inverse r for q. Then rs is a two-sided homotopy inverse for $f = pq$.

Conversely, suppose f has a homotopy inverse. Again factor f as

$$A \underset{\sim}{\overset{q}{\rightarrowtail}} C \overset{p}{\twoheadrightarrow} X.$$

We want to show that p is a weak equivalence and then we'll be done by 2-of-3.

Let $g : X \to A$ be a homotopy inverse to f, and let $H : \mathrm{cyl}(X) \to X$ be a homotopy from fg to id_X. Since X is cofibrant, $i_0 : X \overset{\sim}{\rightarrowtail} \mathrm{cyl}(X)$ is an acyclic cofibration. Lift H to $\tilde{H}$ in the diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & A & \underset{\sim}{\overset{q}{\rightarrowtail}} & C \\ {\scriptstyle i_0} \big\downarrow \sim & & \overset{\tilde{H}}{\nearrow} & & \big\downarrow p \\ \mathrm{cyl}(X) & & \xrightarrow[H]{} & & X. \end{array}$$

Set $s = \tilde{H} i_1$. Then $ps = \mathrm{id}_X$ and $\tilde{H}$ is a homotopy $qg \sim s$.

Use the first part of this proof to produce a homotopy inverse r for q. Then

$$pq = f \quad \Rightarrow \quad pqr = fr \quad \Rightarrow \quad p \sim fr$$

and so

$$sp \sim qgp \sim qgfr \sim qr \sim \mathrm{id}_C.$$

We claim that it follows from this that sp is a weak equivalence. Let K be a left homotopy from id_C to sp. Since $\mathrm{id}_C = Kj_0$ and j_0 are weak equivalences, so is K. Then because j_1 is a weak equivalence, so is $Kj_1 = sp$.

Finally, observe that p is a retract of sp, so it too is a weak equivalence.

$$\begin{array}{ccccc} C & = & C & = & C \\ {\scriptstyle p}\downarrow & & \sim\downarrow{\scriptstyle sp} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[s]{} & C & \xrightarrow[p]{} & X. \end{array}$$

□

4. The homotopy category of a model category

For each object X in $\mathbf{C}$, we can apply factorization to $\emptyset \to X$ to obtain an acyclic fibration $QX \overset{\sim}{\twoheadrightarrow} X$ with QX cofibrant. Dually, we can also apply factorization to $X \to *$ to obtain an acyclic cofibration $X \overset{\sim}{\rightarrowtail} RX$ with RX fibrant. We shall call QX (resp. RX) the *cofibrant* (resp. *fibrant*) *replacement* of X.

Example 4.1. In the projective model structure on $\mathbf{Ch}_R$, a cofibrant resolution is precisely a projective resolution/cover from homological algebra. A fibrant resolution in the injective model structure is an injective resolution/envelope.

Definition 4.2. The *homotopy category* $\mathrm{Ho}(\mathbf{C})$ is the category with the same objects as $\mathbf{C}$ and

$$\mathrm{Ho}(\mathbf{C})(X, Y) = \pi(RQX, RQY).$$

There is a natural functor $\gamma : \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$.

Proposition 4.3 ([DS95, Prop. 5.8])**.** *If f is a morphism of $\mathbf{C}$, then $\gamma(f)$ is an isomorphism in* $\mathrm{Ho}(\mathbf{C})$ *iff f is a weak equivalence.*

In fact we can say something stronger:

Theorem 4.4 ([DS95, Thm. 6.2])**.** *The functor $\gamma : \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ is a localization of $\mathbf{C}$ with respect to its weak equivalences $\mathbf{W}$. That is to say, γ is initial amongst functors that takes weak equivalences in $\mathbf{C}$ to isomorphisms in* $\mathrm{Ho}(\mathbf{C})$.

This achieves one of the purpose of model categories stated in the introduction.

5. Derived functors

Recall:

Definition 5.1. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between model categories. A *total left derived functor* $\mathbb{L}F : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ that is equipped with a terminal natural transformation $\mathbb{L}F \circ \gamma_{\mathbf{C}} \to \gamma_{\mathbf{D}} \circ F$.

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\ {\scriptstyle \gamma_{\mathbf{C}}}\downarrow & \Uparrow & \downarrow{\scriptstyle \gamma_{\mathbf{D}}} \\ \mathrm{Ho}(\mathbf{C}) & \xrightarrow[\mathbb{L}F]{} & \mathrm{Ho}(\mathbf{D}) \end{array}$$

Dually, a *total right derived functor* $\mathbb{R}F : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ that is equipped with an initial natural transformation $\gamma_{\mathbf{D}} \circ F \to \mathbb{R}F \circ \gamma_{\mathbf{C}}$.

We return to a question asked in the introduction: what conditions are needed on a functor $F : \mathbf{C} \to \mathbf{D}$ so that it gives a nice derived functor $\mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$? We said previously that if F were already homotopical, then it has a derived functor. It turns out that there is a larger class of functors that possess nicely-behaved derived functors.

Definitions 5.2. Let $\mathbf{C}$ and $\mathbf{D}$ be model categories. A functor $F : \mathbf{C} \to \mathbf{D}$ is

(i) a *left Quillen functor* if F is a left adjoint and preserves **cof** and $\mathbf{W} \cap \mathbf{cof}$, and
(ii) a *right Quillen functor* if F is a right adjoint and preserves **fib** and $\mathbf{W} \cap \mathbf{fib}$.

Lemma 5.3 (Ken Brown's lemma)**.** *Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between model categories. If F carries acyclic cofibrations between cofibrant objects to weak equivalences, then F preserves all weak equivalences between cofibrant objects.*

Exercise 5.4. Prove this!

As a consequence, a left Quillen functor preserves weak equivalences between cofibrant objects. Dually, a right Quillen functor preserves weak equivalences between fibrant objects.

Now we see how to define derived functors. Loosely speaking, $\mathbb{L}F$ is given by first taking a cofibrant replacement and then applying F, and $\mathbb{R}F$ is fibrant replacement followed by F.

Theorem 5.5 (Quillen adjunction/equivalence theorem)**.** *Suppose $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ form a* Quillen adjunction*, i.e., $F \dashv G$ and F is a left Quillen functor ($\Leftrightarrow$ G is a right Quillen functor). Then the total derived functors $\mathbb{L}F : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ and $\mathbb{R}G : \mathrm{Ho}(\mathbf{D}) \to \mathrm{Ho}(\mathbf{C})$ exist and form an adjoint pair on the homotopy categories.*

Suppose additionally that (F, G) form a Quillen equivalence*: i.e., whenever A is cofibrant and X is fibrant, a map $f : A \to GX$ is a weak equivalence iff its adjoint $f^\flat : FA \to X$ is a weak equivalence. Then $\mathbb{L}F$ and $\mathbb{R}G$ are inverse equivalences of homotopy categories.*

Proof. Use the previous lemma. See [DS95, Thm. 9.7]. □

Examples 5.6.

(1) Consider the category $\mathbf{Ch}_R$ of chain complexes of R-modules and chain maps. Fix a chain complex M and consider the adjunction
$$M \otimes_R ? : \mathbf{Ch}_R \rightleftharpoons \mathbf{Ch}_R : \mathrm{Hom}_R(M, ?).$$
If we equip $\mathbf{Ch}_R$ with the projective model structure on the left and the injective model structure on the right, then this is a Quillen adjunction and gives rise to the derived tensor product $M \otimes^{\mathbb{L}}_R ?$ and the derived hom $\mathbb{R}\,\mathrm{Hom}_R(M, ?)$. Taking homology recovers the usual Tor and Ext.

(2) $|-| : \mathbf{sSet} \leftrightharpoons \mathbf{Spaces} : \mathrm{Sing}$ is a Quillen equivalence. The slogan is "the homotopy theory of topological spaces is the same as the homotopy theory of simplicial sets".

(3) The category of nonnegatively graded chain complexes of abelian groups with the projective model structure is Quillen equivalent to the category of simplicial abelian groups with the standard model structure via the nerve and realization functors. This is the *Dold-Kan correspondence*.

Example 5.7 (Homology is the left derived functor of abelianization)**.** Given a category $\mathbf{C}$, we say that an object X is *abelian* if $\mathbf{C}(-, X)$ is an abelian group. Let $\mathbf{C}_{ab}$ be the full subcategory of abelian objects. Now suppose that $\mathbf{C}$ is a model category and that $\mathbf{C}_{ab}$ is also a model category with the same weak equivalences and fibrations. Moreover, assume that the forgetful functor $U : \mathbf{C}_{ab} \to \mathbf{C}$ has a left adjoint Ab, called *abelianization*, so that (Ab, U) is a Quillen pair.

We define *Quillen homology*[7] to be the left derived functor of Ab.

For concreteness, specialize to the case where $\mathbf{C} = \mathbf{Spaces}$ (or $\mathbf{sSet}$). Then we have $\mathrm{Ab}(X) = SP(X)$, the infinite symmetric product of X. Note that
$$\pi_* \mathbb{L}\,\mathrm{Ab}(X) \cong \pi_*(SP(X)) \cong \tilde{H}_*(X; \mathbb{Z}).$$
The last isomorphism is the *Dold-Thom theorem*. So "homology is the left derived functor of abelianization".

You can also recover other types of homology or cohomology from this.

[7] Possible paper topic # 14.

Here is a high-level summary of what we've "accomplished":

Theorem 5.8 ([Hov99, Thm. 1.4.3])**.** *The homotopy category, derived adjunction, and derived natural transformation define a pseudo-2-functor* $\mathrm{Ho} : \mathbf{ModelCat} \to \mathbf{Cat}_{\mathrm{ad}}$ *from the 2-category of model categories and Quillen adjunctions to the 2-category of categories and adjunctions.*

5.1. **Homotopy limits and colimits.** In this section we discuss an application of the theory of model categories: defining homotopically correct common operations on objects.

Example 5.9. Consider the two horizontal maps in the following commutative square

$$\begin{array}{ccc} S^{n-1} = \partial D^n & \hookrightarrow & D^n \\ \| & & \downarrow{\sim} \\ S^{n-1} & \longrightarrow & * \end{array}$$

All the vertical arrows are homotopy equivalences, but the cofibers of the horizontal maps are quite different. This shows that taking cofibers – in general, any limit or colimit – is not homotopically well-behaved. Intuitively, the quotient – where we demand points be identified on the nose – is too homotopy-insensitive; instead, one ought to instead "join" points with paths. This gives

$$\mathrm{colim}(S^{n-1} \leftarrow S^{n-1} \times I \to *) \cong S^n.$$

Thus, according to this reasoning, the cofiber of the top map, i.e., S^n, is the "true" cofiber. Model categories gives a way to make this intuition precise: only the top map is a cofibration.

Homotopy limits and colimits can be constructed as derived functors in the setting of model categories. Let $\mathbf{C}$ be a model category. Recall we have adjunctions $\mathrm{colim} \vdash c \vdash \lim$, where c is the constant diagram functor. For concreteness, let $\mathbf{J}$ be a small indexing category over which we want to take a colimit. To order to apply the Quillen adjunction theorem, we need to find a model structure on $\mathbf{C}^{\mathbf{J}}$ so that $\mathrm{colim} \vdash c$ is a Quillen pair. Since we want c to be a right Quillen functor, we see that $\mathbf{C}^{\mathbf{J}}$ ought to have pointwise fibrations and pointwise weak equivalences.

Proposition 5.10. *If* $\mathbf{J}$ *is* Reedy *(e.g., coequalizer, pushout, and direct limit diagrams are all Reedy), there is a model structure on* $\mathbf{C}^{\mathbf{J}}$ *called the* Reedy model structure *whose weak equivalences are precisely pointwise weak equivalences.*

This model structure is often suitable for computing homotopy colimits or homotopy limits. (The precise condition is $\mathbf{J}$ *having* cofibrant constants *or* fibrant constants *respectively.)*

Using this, one can define, e.g., homotopy colimits as

$$\mathrm{hocolim} = \mathbb{L}\,\mathrm{colim} = \mathrm{colim} \circ Q,$$

i.e., by taking a cofibrant resolution of the diagram and then taking the usual colimit.

Unravelling the definition of the Reedy model structure in the case $\mathbf{J} = [1]$, we see $X : [1] \to \mathbf{C}$ is a *Reedy cofibrant diagram* if X_0 and X_1 are cofibrant and $X_0 \to X_1$ is a cofibration.

So roughly speaking in general, cofibrant replacement replaces all the objects with cofibrant ones and all the arrows in the diagram with cofibrations. This accords with our philosophy that the cofiber of $S^{n-1} \to *$ should be S^{n-1} as in the example.

Example 5.11. Examples of homotopy colimits include mapping cones, mapping cylinders, mapping telescopes.

If the indexing category $\mathbf{J}$ turns out to not be Reedy, we can put assumptions on the model category $\mathbf{C}$ instead:

Remark 5.12. Let $\mathbf{J}$ be any small category.

(a) If $\mathbf{C}$ is a *cofibrantly generated* model category, then $\mathbf{C}^{\mathbf{J}}$ has the *projective model structure*, which is suitable for computing homotopy colimits.
  Most model categories in practice are cofibrantly generated.
(b) If $\mathbf{C}$ is *combinatorial*, then $\mathbf{C}^{\mathbf{J}}$ has the *injective model structure*, which is suitable for computing homotopy limits.

(c) It is interesting to note that the Reedy model structure sits between the projective and injective model structures, when all three exist. In other words, the identity maps give Quillen equivalences

$$\mathbf{C}^{\mathbf{J}}_{\text{proj}} \xrightarrow{\sim} \mathbf{C}^{\mathbf{J}}_{\text{Reedy}} \xrightarrow{\sim} \mathbf{C}^{\mathbf{J}}_{\text{inj}}.$$

Remark 5.13. There is an alternative approach to homotopy colimits (resp. limits) if we work with simplicial (resp. cosimplicial) objects, using the bar (resp. cobar) construction. In good cases the two answers agree. The idea is that the simplicial (resp. cosimplicial) replacement of a diagram of cofibrant (resp. fibrant) objects is automatically Reedy cofibrant (resp. Reedy fibrant).

Also see: the *Bousfield-Kan formula.*

6. The small object argument

Often, the hardest part of equipping a category with a model structure is to prove factorization. This is often done using Quillen's small object argument.

Theorem 6.1 (Small object argument)**.** *Let* $I \subset \mathrm{Mor}(\mathbf{C})$ *be a collection of morphisms such that* $\mathbf{C}$ *has all colimits. Suppose for each* $f \in I$*,* $\mathrm{dom}(f)$ *is small (e.g., if* $\mathbf{C}$ *is locally presentable). Then* $(\mathrm{cell}(I), \mathrm{rlp}(I))$ *is a weak factorization system.*

I can't be bothered to finish typing this out…it's unlikely I'll get this far anyway.

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