Homework #4

Math 231b

"Due": April 12th, 2017

Guidelines:

- Type up your solution to the assignment in LaTeX. (You might want to avail yourself of the excellent diagrams package `tikz-cd`.)
- Submit the PDF via Canvas, in the Assignments section.

Failure to meet these guidelines may result in loss of points.[1]

Problem 1. Prove the transgressive differential lemma from class. Let $F \xrightarrow{i} E \to B$ be a fibration, and let

$$B \xleftarrow{\pi} C(i) \xrightarrow{\delta} \Sigma F$$

by the naturally induced maps. Show that the following situations are equivalent:

- A class $x \in H_n B$ has $d_{<n}(x) = 0$ and $d_n(x) = y$ for some class $y \in H_{n-1} F$ (up to some indeterminacy).
- There is a class $\tau(x) \in H_n C(i)$ with $\delta_* \tau x = y$ and $\pi_* \tau x = x$.

Problem 2. The *(2–adic) Bockstein spectral sequence* is the filtration spectral sequence arising from the diagram

$$\begin{tikzcd}
\mathbb{Z}_2^\wedge \arrow[r] & \cdots \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[d] \\
 & \cdots & \mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2.
\end{tikzcd}$$

Applying $H^*(X; -)$ to this diagram of coefficients gives a spectral sequence of signature

$$E_1^{*,*} = \bigoplus_* H^*(X; \mathbb{F}_2) \cong \mathbb{F}_2[w] \otimes H^*(X; \mathbb{F}_2) \Rightarrow H^*(X; \mathbb{Z}_2^\wedge),$$

where the E_1–page consists of many duplicated copies of $H^*(X; \mathbb{F}_2)$, which we can think of as tagged by monomials in w.[2]

1. Show that the differentials in this spectral sequence are "w–linear", i.e., $d_r^{\mathrm{BSS}}(w^k x) = w^k d_r^{\mathrm{BSS}}(x)$.
2. Show that a torsion-free class $x \in H^*(X; \mathbb{Z}_2^\wedge)$ is in $\ker d_r^{\mathrm{BSS}}$ on all pages E_r and never in $\operatorname{im} d_r^{\mathrm{BSS}}$. Demonstrate that this condition is equivalent to the corresponding class in the spectral sequence being w–torsion–free.
3. More generally, show that the order of w–torsion of a class on the E_∞ page of the spectral sequence is identical to the 2–primary torsion order of the corresponding cohomology class in $H^*(X; \mathbb{Z}_2^\wedge)$.

[1] This version of the assignment was compiled on March 27, 2017.
[2] There is (of course) also a homological version of this construction, which you should also be aware of.

4. Show that d_1^{BSS} in this spectral sequence is computed by the Steenrod square Sq^1.

Problem 3. Use this spectral sequence to make a calculation of $H^*(K(\mathbb{Z}/2,2);\mathbb{Z}_2^\wedge)$ from the calculation of $H^*(K(\mathbb{Z}/2,2);\mathbb{F}_2)$ given in class. You will want to know the following mysterious formula:[3] for any class $x \in H^{\mathrm{even}}(X;\mathbb{F}_2)$ where $d_r^{\mathrm{BSS}}(x)$ is defined, we have

$$d_r^{\mathrm{BSS}}(x^2) = \begin{cases} \mathrm{Sq}^1(x) \cdot x + \mathrm{Sq}^{|x|}\,\mathrm{Sq}^1(x) & \text{for } r = 2, \\ d_{r-1}^{\mathrm{BSS}}(x) \cdot x & \text{for } r > 2. \end{cases}$$

Problem 4. Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fiber sequence, let $u \in H^n(F;\mathbb{F}_2)$ be class that transgresses to $\tau(u) \in H^{n+1}(B;\mathbb{F}_2)$, and suppose that for some integer $i \geq 1$ there is a Bockstein differential $d_i^{\mathrm{BSS}} v = \tau(u)$. Show that $d_{i+1}^{\mathrm{BSS}} p^* v$ is then defined and that $j^* d_{i+1}^{\mathrm{BSS}} p^*(v) = d_1^{\mathrm{BSS}}(u)$, where again d_1^{BSS} is the first Bockstein differential.[4]

Problem 5. I may not have done a good job of stating this problem. If you run into issues with solving this, please email me so that I can fix whatever mistakes I've made. (The algebra extensions in part 2 seem particularly fishy. . . .) In this problem, you will reinvent one of the main results of unstable rational homotopy. For a simply connected space X, we inductively define its *rationalization* to be a space $\mathbb{Q} \otimes X$ under X as follows: given a Postnikov fibration

$$K(\pi_n X, n) \to X[0,n] \to X[0,n),$$

and the rationalization map $X[0,n) \to (\mathbb{Q} \otimes X)[0,n)$, we construct a corresponding Postnikov fibration for $\mathbb{Q} \otimes X$ as the back face in

$$\begin{tikzcd}
 & K(\mathbb{Q}\otimes\pi_n X, n) \arrow[rr, equal] \arrow[dd] & & K(\mathbb{Q}\otimes\pi_n X, n) \arrow[dd] \\
K(\pi_n X, n) \arrow[ur] \arrow[rr, equal] \arrow[dd] & & K(\pi_n X, n) \arrow[ur] \arrow[dd] & \\
 & (\mathbb{Q}\otimes X)[0,n] \arrow[rr] \arrow[dd] & & * \arrow[dd] \\
X[0,n] \arrow[ur] \arrow[rr] \arrow[dd] & & * \arrow[ur] \arrow[dd] & \\
 & (\mathbb{Q}\otimes X)[0,n) \arrow[rr] & & K(\mathbb{Q}\otimes\pi_n X, n+1) \\
X[0,n) \arrow[ur] \arrow[rr] & & K(\pi_n X, n+1). \arrow[ur] &
\end{tikzcd}$$

Here the nodes $X[0,n]$ and $(\mathbb{Q}\otimes X)[0,n]$ are *defined* as the total spaces of the pullback fibrations, and the map between them is induced by the universal map of fibrations. We set $\mathbb{Q}\otimes X$ to be the homotopy inverse limit

$$\mathbb{Q}\otimes X = \lim_n (\mathbb{Q}\otimes X)[0,n],$$

which has the factorization property

$$\begin{tikzcd}
\pi_* X \arrow[r] \arrow[rr, bend left] & \mathbb{Q}\otimes\pi_* X \arrow[r, "\simeq"] & \pi_*(\mathbb{Q}\otimes X).
\end{tikzcd}$$

[3] This is Proposition 6.8 of May's *A general algebraic approach to Steenrod operations*.

[4] I haven't actually tried to work this out. You might find it helpful to know that there's a re-indexing of the Bockstein spectral sequence, where you instead use the inverse system $\{\mathbb{Z}/(2^j)\}_{j=1}^{\infty}$ and identify all the *fibers* of these maps as $\mathbb{Z}/2$—or maybe not.

Now, justify the following claims:

1. The rational cohomology of rational Eilenberg–Mac Lane spaces is given by
$$H^*(K(\mathbb{Q},n);\mathbb{Q}) = \begin{cases} \mathbb{Q}[x_n] & \text{if } n \text{ is even,} \\ \mathbb{Q}[x_n]/x_n^2 & \text{if } n \text{ is odd.} \end{cases}$$
2. The cohomology $H^*(X(n,\infty);\mathbb{Q})$ *as well as its ring structure* are completely determined by the cohomology ring $H^*(X[n,\infty);\mathbb{Q})$.
3. The map $X \to \mathbb{Q} \otimes X$ is an isomorphism on rational cohomology.
4. The Postnikov fibrations $K(\mathbb{Q} \otimes \pi_n X, n) \to (\mathbb{Q} \otimes X)[0,n] \to (\mathbb{Q} \otimes X)[0,n)$ give a model for $C^*(X;\mathbb{Q})$ whose underlying graded-commutative algebra is *free* and which uses the minimal number of algebra generators.[5]
5. Any rational commutative differential-graded algebra A^* with $A^0 = \mathbb{Q}$ and $A^1 = 0$ inductively receives a quasi-isomorphism from a Sullivan model.[6][7]
6. There is a sequence of Postnikov sections $X[0,n) \to K(\pi_n X, n+1)$, hence a space X, whose Sullivan model is the one associated to A^*.
7. Given a Sullivan model for $C^*(X;\mathbb{Q})$, its indecomposables compute the rational homotopy groups of X.
8. The rational homotopy groups of S^n, $n > 1$, are given by
$$\mathbb{Q} \otimes \pi_* S^n = \begin{cases} \Sigma^n \mathbb{Q} & \text{if } n \text{ is odd,} \\ \Sigma^n \mathbb{Q} \oplus \Sigma^{2n-1}\mathbb{Q} & \text{if } n \text{ is even.} \end{cases}$$

Problem 6.

1. The tensor product of line bundles induces a map
$$BU(1) \times BU(1) \xrightarrow{\otimes} BU(1)$$
on the object $BU(1)$ representing the functor $X \mapsto \{\text{iso-classes of line bundles on } X\}$. Describe the behavior of this map in ordinary cohomology with $\mathbb{Z}$ coefficients.
2. In general, the tensor product of vector bundles induces a similar map
$$BU(n) \times BU(m) \xrightarrow{\otimes} BU(nm).$$
Describe the behavior of this map in ordinary cohomology as well.

Problem 7. The dual Steenrod algebra is a *Hopf algebra*, meaning that it not only has a multiplication map but also a diagonal map $\Delta\colon \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ and an antipode map $\chi\colon \mathcal{A}_* \to \mathcal{A}_*$. In class, we deduced a formula for Δ, and we showed that as an algebra the dual Steenrod algebra forms a polynomial ring. The antipode fits into the commutative diagram

[5] Such a presentation of the rational cochain complex is called a *Sullivan minimal model*. It may please you to check that two such models are related by a chain homotopy equivalence.

[6] In fact, this happens in a natural way: a map $A^* \to B^*$ of cDGAs induces a map of their Sullivan models.

[7] It's fun / instructive to see the natural algorithm for this *fail* in the case of $C^*(S^1 \vee S^1;\mathbb{Q})$.

$$
\begin{tikzcd}
& \mathcal{A}\otimes\mathcal{A} \arrow[rr, "\chi\otimes 1"] & & \mathcal{A}\otimes\mathcal{A} \arrow[dr, "\mu"] & \\
\mathcal{A} \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"] & & \mathbb{F}_2 \arrow[rr, "\eta"] & & \mathcal{A} \\
& \mathcal{A}\otimes\mathcal{A} \arrow[rr, "1\otimes\chi"] & & \mathcal{A}\otimes\mathcal{A}, \arrow[ur, "\mu"] &
\end{tikzcd}
$$

along with the algebra unit η and counit ε. Use all this to give a recursive formula for the behavior of χ.