Homework #4

Math 231b

"Due": April 12th, 2017

Guidelines:

- Type up your solution to the assignment in IATEX. (You might want to avail yourself of the excellent diagrams package tikz-cd.)
- Submit the PDF via Canvas, in the Assignments section.

Failure to meet these guidelines may result in loss of points.¹

Problem 1. Prove the transgressive differential lemma from class. Let $F \xrightarrow{i} E \to B$ be a fibration, and let

$$B \xleftarrow{\pi} C(i) \xrightarrow{\delta} \Sigma F$$

by the naturally induced maps. Show that the following situations are equivalent:

- A class $x \in H_n B$ has $d_{< n}(x) = 0$ and $d_n(x) = y$ for some class $y \in H_{n-1} F$ (up to some indeterminacy).
- There is a class $\tau(x) \in H_nC(i)$ with $\delta_*\tau x = y$ and $\pi_*\tau x = x$.

Problem 2. The (2-adic) Bockstein spectral sequence is the filtration spectral sequence arising from the diagram

Applying $H^*(X; -)$ to this diagram of coefficients gives a spectral sequence of signature

$$E_1^{*,*} = \bigoplus_* H^*(X; \mathbb{F}_2) \cong \mathbb{F}_2[w] \otimes H^*(X; \mathbb{F}_2) \Rightarrow H^*(X; \mathbb{Z}_2^{\wedge}),$$

where the E_1 -page consists of many duplicated copies of $H^*(X; \mathbb{F}_2)$, which we can think of as tagged by monomials in w.²

- 1. Show that the differentials in this spectral sequence are "w-linear", i.e., $d_r^{\mathrm{BSS}}(w^k x) = w^k d_r^{\mathrm{BSS}}(x)$.
- 2. Show that a torsion-free class $x \in H^*(X; \mathbb{Z}_2^{\wedge})$ is in $\ker d_r^{\mathrm{BSS}}$ on all pages E_r and never in $\operatorname{im} d_r^{\mathrm{BSS}}$. Demonstrate that this condition is equivalent to the corresponding class in the spectral sequence being w-torsion-free.
- 3. More generally, show that the order of w-torsion of a class on the E_{∞} page of the spectral sequence is identical to the 2-primary torsion order of the corresponding cohomology class in $H^*(X; \mathbb{Z}_2^{\wedge})$.

¹This version of the assignment was compiled on March 27, 2017.

²There is (of course) also a homological version of this construction, which you should also be aware of.

4. Show that d_1^{BSS} in this spectral sequence is computed by the Steenrod square Sq^1 .

Problem 3. Use this spectral sequence to make a calculation of $H^*(K(\mathbb{Z}/2,2);\mathbb{Z}_2^{\wedge})$ from the calculation of $H^*(K(\mathbb{Z}/2,2);\mathbb{F}_2)$ given in class. You will want to know the following mysterious formula:³ for any class $x \in H^{\text{even}}(X;\mathbb{F}_2)$ where $d_r^{\text{BSS}}(x)$ is defined, we have

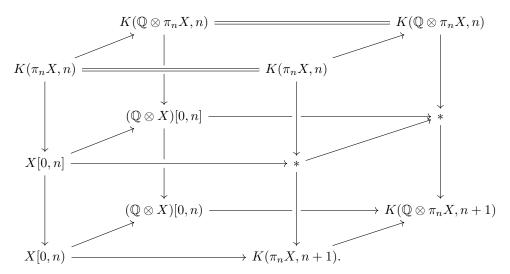
$$d_r^{\mathrm{BSS}}(x^2) = \begin{cases} \operatorname{Sq}^1(x) \cdot x + \operatorname{Sq}^{|x|} \operatorname{Sq}^1(x) & \text{for } r = 2, \\ d_{r-1}^{\mathrm{BSS}}(x) \cdot x & \text{for } r > 2. \end{cases}$$

Problem 4. Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fiber sequence, let $u \in H^n(F; \mathbb{F}_2)$ be class that transgresses to $\tau(u) \in H^{n+1}(B; \mathbb{F}_2)$, and suppose that for some integer $i \geq 1$ there is a Bockstein differential $d_i^{\mathrm{BSS}}v = \tau(u)$. Show that $d_{i+1}^{\mathrm{BSS}}p^*v$ is then defined and that $j^*d_{i+1}^{\mathrm{BSS}}p^*(v) = d_1^{\mathrm{BSS}}(u)$, where again d_1^{BSS} is the first Bockstein differential.

Problem 5. I may not have done a good job of stating this problem. If you run into issues with solving this, please email me so that I can fix whatever mistakes I've made. (The algebra extensions in part 2 seem particularly fishy....) In this problem, you will reinvent one of the main results of unstable rational homotopy. For a simply connected space X, we inductively define its rationalization to be a space $\mathbb{Q} \otimes X$ under X as follows: given a Postnikov fibration

$$K(\pi_n X, n) \to X[0, n] \to X[0, n),$$

and the rationalization map $X[0,n) \to (\mathbb{Q} \otimes X)[0,n)$, we construct a corresponding Postnikov fibration for $\mathbb{Q} \otimes X$ as the back face in



Here the nodes X[0,n] and $(\mathbb{Q} \otimes X)[0,n]$ are defined as the total spaces of the pullback fibrations, and the map between them is induced by the universal map of fibrations. We set $\mathbb{Q} \otimes X$ to be the homotopy inverse limit

$$\mathbb{Q} \otimes X = \lim_{n} (\mathbb{Q} \otimes X)[0, n],$$

which has the factorization property

$$\pi_*X \xrightarrow{\cong} \mathbb{Q} \otimes \pi_*X \xrightarrow{\cong} \pi_*(\mathbb{Q} \otimes X).$$

 $^{^3{\}rm This}$ is Proposition 6.8 of May's A general algebraic approach to Steenrod operations.

⁴I haven't actually tried to work this out. You might find it helpful to know that there's a re-indexing of the Bockstein spectral sequence, where you instead use the inverse system $\{\mathbb{Z}/(2^j)\}_{j=1}^{\infty}$ and identify all the *fibers* of these maps as $\mathbb{Z}/2$ —or maybe not.

Now, justify the following claims:

1. The rational cohomology of rational Eilenberg–Mac Lane spaces is given by

$$H^*(K(\mathbb{Q}, n); \mathbb{Q}) = \begin{cases} \mathbb{Q}[x_n] & \text{if } n \text{ is even,} \\ \mathbb{Q}[x_n]/x_n^2 & \text{if } n \text{ is odd.} \end{cases}$$

- 2. The cohomology $H^*(X(n,\infty);\mathbb{Q})$ as well as its ring structure are completely determined by the cohomology ring $H^*(X[n,\infty);\mathbb{Q})$.
- 3. The map $X \to \mathbb{Q} \otimes X$ is an isomorphism on rational cohomology.
- 4. The Postnikov fibrations $K(\mathbb{Q} \otimes \pi_n X, n) \to (\mathbb{Q} \otimes X)[0, n] \to (\mathbb{Q} \otimes X)[0, n)$ give a model for $C^*(X; \mathbb{Q})$ whose underlying graded-commutative algebra is *free* and which uses the minimal number of algebra generators.⁵
- 5. Any rational commutative differential-graded algebra A^* with $A^0 = \mathbb{Q}$ and $A^1 = 0$ inductively receives a quasi-isomorphism from a Sullivan model.⁶⁷
- 6. There is a sequence of Postnikov sections $X[0,n) \to K(\pi_n X, n+1)$, hence a space X, whose Sullivan model is the one associated to A^* .
- 7. Given a Sullivan model for $C^*(X;\mathbb{Q})$, its indecomposables compute the rational homotopy groups of X.
- 8. The rational homotopy groups of S^n , n > 1, are given by

$$\mathbb{Q} \otimes \pi_* S^n = \begin{cases} \Sigma^n \mathbb{Q} & \text{if } n \text{ is odd,} \\ \Sigma^n \mathbb{Q} \oplus \Sigma^{2n-1} \mathbb{Q} & \text{if } n \text{ is even.} \end{cases}$$

Problem 6. 1. The tensor product of line bundles induces a map

$$BU(1) \times BU(1) \xrightarrow{\otimes} BU(1)$$

on the object BU(1) representing the functor $X \mapsto \{\text{iso-classes of line bundles on } X\}$. Describe the behavior of this map in ordinary cohomology with \mathbb{Z} coefficients.

2. In general, the tensor product of vector bundles induces a similar map

$$BU(n) \times BU(m) \xrightarrow{\otimes} BU(nm).$$

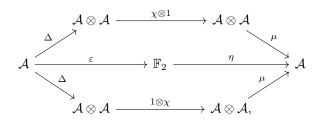
Describe the behavior of this map in ordinary cohomology as well.

Problem 7. The dual Steenrod algebra is a *Hopf algebra*, meaning that it not only has a multiplication map but also a diagonal map $\Delta \colon \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ and an antipode map $\chi \colon \mathcal{A}_* \to \mathcal{A}_*$. In class, we deduced a formula for Δ , and we showed that as an algebra the dual Steenrod algebra forms a polynomial ring. The antipode fits into the commutative diagram

⁵Such a presentation of the rational cochain complex is called a *Sullivan minimal model*. It may please you to check that two such models are related by a chain homotopy equivalence.

⁶In fact, this happens in a natural way: a map $A^* \to B^*$ of cDGAs induces a map of their Sullivan models.

⁷It's fun / instructive to see the natural algorithm for this fail in the case of $C^*(S^1 \vee S^1; \mathbb{Q})$.



along with the algebra unit η and counit ε . Use all this to give a recursive formula for the behavior of χ .