Homework #3

Math 231b

"Due": March 22nd, 2017

Guidelines:

- Type up your solution to the assignment in LaTeX. (You might want to avail yourself of the excellent diagrams package `tikz-cd`.)
- Submit the PDF via Canvas, in the Assignments section.

Failure to meet these guidelines may result in loss of points.[1]

Problem 1. Consider a diagram of three inverse systems of abelian groups

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A_{n+1} & \xrightarrow{f_{n+1}} & A_n & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & B_{n+1} & \xrightarrow{g_{n+1}} & B_n & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{h_{n+1}} & C_n & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots ,
\end{array}$$

such that every column forms a short exact sequence.

1. Show that the limit of, say, (A_n) can be described by the kernel sequence
$$\lim A_n \xrightarrow{\ker} \prod_n A_n \xrightarrow{\prod_n \mathrm{id} - \prod_n f_n} \prod_n A_n.$$
2. The map on limits $\lim A_n \to \lim B_n \to \lim C_n$ no longer need be short-exact. Define $\lim{}^1 A_n$ to be the *cokernel* of the map described above, and show that there is instead an exact sequence of the form
$$0 \to \lim A_n \to \lim B_n \to \lim C_n \to \lim{}^1 A_n \to \lim{}^1 B_n \to \lim{}^1 C_n \to 0.$$

Problem 2. Consider a tower of fibrations

$$\cdots \to X_{n+1} \xrightarrow{f_{n+1}} X_n \to \cdots .$$

[1]This version of the assignment was compiled on March 22, 2017.

Show that there is a short exact sequence, called the *Milnor sequence*, given by

$$0 \to \lim_n{}^1 (\pi_{m+1} X_n) \to \pi_m \left(\lim_n X_n\right) \to \lim_n (\pi_m X_n) \to 0.$$

(Hint: find a model for the limit of the tower of fibrations analogous to the one for the colimit of a tower of inclusions presented in Switzer 7.53. This model is itself inspired by rewriting the tower as an endomorphism of an infinite product, and filtering this model by "near the start" and "near the end" of the endomorphism.)

Problem 3. [2] Produce the Milnor short exact sequence for a generalized cohomology theory E applied to an increasing union of spaces X_n:

$$0 \to \lim_n{}^1 (E^{m-1} X_n) \to E^m \left(\operatorname{colim}_n X_n\right) \to \lim_n (E^m X_n) \to 0.$$

Problem 4.

1. Use this to calculate the integral cohomology of "the circle with p inverted". This space is given by infinitely iterating the mapping cylinder construction on the p–fold covering

$$S^1 \xrightarrow{p} S^1.$$

   (That is: the first stages of this look like $S^1 \cup_p (S^1 \times I)$, then $(S^1 \cup_p (S^1 \times I)) \cup_p (S^1 \times I)$, $\ldots$.)[3]

2. Compare your answer with the *homology* of this same space and analyze the behavior of the universal coefficient sequence.

Task 5. Read pages 158–163 of Switzer, which describe the representability of sheaf-like functors defined only on *finite* CW-complexes. (In particular, this makes fairly intensive use of the understanding of inverse limits which you have just developed.)

Task 6. Strongly consider reading pages 346–351 of Switzer, which actually goes through the identification of the Serre E^2 term. It's very tedious, but it's worth seeing once. Alternatively, you could these course notes, which give a much prettier description of the Serre spectral sequence in terms of a double complex: `http://math.mit.edu/classes/18.906/spr09/sss.pdf` . As trade, you then have to additionally work out how such a spectral sequence arises as a filtration spectral sequence.

Problem 7. As in class, define $\mathbb{S}_{(p)}$ to be the spectrum representing $X \mapsto h\mathsf{Spectra}(\Sigma^\infty X, \mathbb{S}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

1. Describe the homology functor associated to $\mathbb{S}_{(p)}$. (Hint: restrict attention to finite complexes X, where $DX = F(X, \mathbb{S})$ defines an involutive dual.)
2. Demonstrate $E_{(p)} \simeq E \wedge \mathbb{S}_{(p)}$ for any spectrum E. In particular, this gives $\mathbb{S}_{(p)} \wedge \mathbb{S}_{(p)} \simeq \mathbb{S}_{(p)}$.
3. Define $\pi_{n,(p)} E = [\mathbb{S}^n_{(p)}, E]$. Show $\pi_{n,(p)} E_{(p)} = \pi_n E_{(p)}$.
4. Conclude the more general adjunction

$$h\mathsf{Spectra}(F, E_{(p)}) \cong h\mathsf{Spectra}(F_{(p)}, E_{(p)}).$$

5. Finally, we can also give a concrete construction of $\mathbb{S}_{(p)}$. Consider the infinite directed system

$$\mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \cdots.$$

   Form the mapping telescope T associated to this system and check that this gives a model for $\mathbb{S}_{(p)}$ in $h\mathsf{Spectra}$.

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[2] This problem can also be done independently of Problem 2, by directly using the endomorphism-cylinder construction presented in Switzer and applying cohomology to that, rather than taking the time to figure out what its dual looks like and remembering that cohomology appears as the homotopy of a certain spectral object.

[3] If you haven't seen this construction before, you should check that for a ring element $r \in R$ and an R–module M,

$$\operatorname{colim}\left(M \xrightarrow{r} M \xrightarrow{r} M \xrightarrow{r} \cdots\right) = M[r^{-1}].$$