Homework #1

Math 231b

"Due": February 15th, 2017

Guidelines:

- Type up your solution to the assignment in LaTeX. (You might want to avail yourself of the excellent diagrams package `tikz-cd`.)
- Submit the PDF via Canvas, in the Assignments section.

Failure to meet these guidelines may result in loss of points.

Problem 1. Consider the unreduced mapping cylinder M_f of a map $f\colon X \to Y$, together with its inclusion $i\colon Y \to M_f$. Show that the usual retraction $r\colon M_f \to Y$ of i (which projects down from the interval coordinate) induces an isomorphism $r_*\colon \pi_* M_f \to \pi_* Y$. (We used this fact when constructing the homotopy long exact sequence of an arbitrary pair.)

Problem 2. Show that if $f\colon X \to Y$ is a homotopy equivalence, then $f_*\colon \pi_*(X, x_0) \to \pi_*(Y, f(x_0))$ is an isomorphism for all choices of $x_0 \in X$ and $* \geq 0$.

Problem 3. In class, I used the mysterious phrase "limit condition" twice. Given a functor $F\colon \mathsf{J} \to \mathsf{C}$, thought of as a J–shaped diagram in C, we define a *cone* of F to be a constant functor $x\colon \mathsf{J} \to \mathsf{C}$ together with a natural transformation $x \to F$. A *limit* of F is a terminal object in the category of cones.

1. Expand the definition of natural transformation and constant functor to reveal that a cone is equivalent to the data of an object $x \in \mathsf{C}$ together with maps $f_j\colon x \to F(j)$ for each object $j \in J$ such that for any map $g\colon j \to j'$ in the diagram there is a commuting triangle

$$\begin{array}{ccc} & x & \\ {}^{f_j}\swarrow & & \searrow^{f_{j'}} \\ F(j) & \xrightarrow{F(g)} & F(j'). \end{array}$$

2. Now expand the definition of limit to see that a limit, expressed as an object ℓ together with maps h_j, has the property that any other cone point x and its maps f_j factor uniquely through a map $x \to \ell$.
3. Show that the product $X \times Y$ is the limit of the diagram with objects X and Y and no non-identity arrows.
4. Show that the equalizer E of a pair of functions $X \rightrightarrows Y$ of sets is indeed the limit of that diagram.

Problem 4. A functor $G\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Sets}$ is called *representable* when there exists an object Y and a natural isomorphism

$$G \xrightarrow{\simeq} \mathsf{Sets}(-, Y).$$

1. From a morphism $t\colon Y \to Y'$ of representing objects, construct a natural transformation $t_*\colon G \to G'$ of the functors they represent.

2. From a natural transformation $G \to G'$ of represented functors, construct a morphism $Y \to Y'$ of the representing objects.

3. Show also that your assignments respect composition of natural transformations and of morphisms.

4. Show that your assignments are mutual inverses, i.e., a natural transformation of representable functors is exactly the same information as a morphism of representing objects.

Congratulations! You have proved the Yoneda lemma: the functor

$$\mathsf{C} \to \mathsf{Categories}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sets})$$

describes a fully faithful embedding.

Problem 5. Explain convincingly why the usual recipe for forming a group structure on $\pi_1(X, x_0)$ does not apply to the set of relative homotopy classes $\pi_1(I, \partial I)$.

Problem 6. Let $p\colon E \to B$ be a map and consider

$$Z = \{(e, \gamma) \in E \times B^I : \; p(e) = \gamma(0)\} \subseteq E \times B^I.$$

A *path lifting function* for p is a map $\lambda\colon Z \to E^I$ with $\lambda(e, \gamma)(0) = e$ and $p \circ \lambda(e, \gamma) = \gamma$.

1. Show that p is a fibration if and only if there is a path lifting function λ for p.

2. Let $p\colon E \to B$ be a fibration with fiber F, and let P_p be the pathspace construction p described in class. Given a path lifting function $\lambda\colon Z \to E^I$ for p, define maps

$$\begin{aligned} g\colon F \to P_p, & \qquad f \mapsto (f, \omega_0), \\ h\colon P_p \to F, & \qquad (e, \gamma) \mapsto [\lambda(e, \gamma^{-1})](1), \end{aligned}$$

   where "γ^{-1}" denotes the path γ run backwards. Show that g and h present the two halves of a homotopy equivalence.