# MATH 130 TERM PAPER INFORMATION

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## GUIDELINES

The goal of these assignments is to get you used to writing mathematical proofs and to organizing them into a coherent document. You'll select a topic from the list below, research it, and synthesize a paper describing it. Your primary goal is to *communicate*, and so the primary test your paper should pass is:

Someone will learn an interesting (and true) thing related to geometry by reading this.

This doesn't necessarily mean copying down fifty fully detailed proofs, nor does it necessarily mean being extremely inventive. It does mean giving a precise proof of *something*, and it also means constructing a coherent narrative around the proof to give it context. This will require you to thoroughly understand the topic for yourself.

I don't expect you to write a perfect paper on your first attempt. For this reason, we will engage in a review process. The cycle for each paper is as follows:

- *Select your topic.* (Midterm: 2/22. Final: 4/11.) Select a topic from the list below. Email me with your selection. You don't have to pick a topic distinct from everyone else in the class.
- *Submit a rough draft.* (Midterm: 3/4. Final: 4/22.) Submit your first draft of the paper for revisions by emailing it to me. I'll read it and return it to you with comments about: correctness, structure, things you've missed that would be helpful to include, examples, ....
- *Submit the final draft.* (Midterm: 3/11. Final: 5/5.) Submit your final draft of the paper for grading by emailing it to me. I'll read it again and return it to you with more comments — and with a score.

All papers should be typed up using the LaTeX typesetting package — not Microsoft Word, not Notepad, not TextEdit, not Pages, not anything else. LaTeX is the de facto standard for written documents in the mathematical world, as well as many other scientific disciplines, so it is good for you to be exposed to it now. There are various tutorials on the internet for installing LaTeX on your home computer, or even for composing documents in the cloud (e.g., Overleaf, ShareLaTeX, ...).

Typesetting geometric diagrams is reasonably tricky. There are two options to consider, if you want to include figures:

(1) Draw the diagrams using some other program, then export them to PDF (or PostScript) and import them using `\includegraphics`. You can read about this here:

`https://en.wikibooks.org/wiki/LaTeX/Importing_Graphics`.

(2) Draw the diagrams inside of LaTeX itself, using a graphics package. This will look much better, but it requires essentially "programming" the diagram, which can be a nightmare. The most robust graphics package is TikZ, and there is a useful respository of examples found at

`http://www.texample.net/tikz/examples/all/`.

**All work should, of course, be your own. Material pulled from external sources should always be properly cited. Material pulled from your classmates' papers is completely prohibited. Acknowledge all help you receive.** (Discussing your paper with your classmates in order to improve it is, of course, encouraged.)

## Midterm topics

3–5 pages is a good goal for the midterm paper. (Please don't feel bound to this. Write the best paper you can, which may not be the paper that conforms to a page length guideline.)

- Completely describe the relationship between Pappas's theorem, Desargues's theorem, and the scissors theorem. (Which implies the other? Which are equivalent? Prove all the equivalences you claim.)
- Use the theory of field extensions (i.e., Galois theory) to describe what number fields are constructible from ruler and compass. Describe the Gauss–Wantzel theorem on constructible polygons. Conclude that constructible angles do not always have constructable trisections.
- Recall the notion of "tangent lines" from calculus. What happens if you try to form approximations of a curve using parabolas? What about using circles? (Match your formula with the definition of the "curvature" of a curve.)
- Develop some of the geometry of complex or quaternionic projective space. What are some interesting theorems that work in $\mathbb{C}\mathrm{P}^2$ but not $\mathbb{R}\mathrm{P}^2$? For instance, how many distinct sorts of conic sections are there in $\mathbb{C}\mathrm{P}^2$?
- In class, we will study "isometries", which are functions which preserve length. They automatically also preserve angle. There is another class of functions which only preserve angle, called "conformal maps". Describe the connection to "holomorphic" functions, and give some interesting examples of conformally equivalent subsets of the plane.
- Describe the version of the Euclidean algorithm that operates by putting curves on a torus into some sort of standard form.
- Morley's trisector theorem and generalizations.
- Rearranging a tetrahedron into a cube (the 3–dimensional generalization of Figure 2.14 of Stillwell.)
- Buffon's needle problem.
- Honeycombs and efficient packing.
- Kepler's laws of planetary motion. (I have a book about this called *Feynman's Lost Lecture* that looks like a reasonable resource.) There's also the geometry of intragalactic planetary motion and early astronomy. (I think that the account in Chapter 2 of `http://link.springer.com/book/10.1007/2F978-3-642-54083-7` is pretty helpful.) There's also also Gallilean geometry. (`http://link.springer.com/book/10.1007/2F978-1-4612-6135-` )
- The game "Set!" and geometry over the finite field $\mathbb{F}_3$. (Some other authors have written short, accessible expositions about this. For instance, see `http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/set.pdf`.)
- **Your own topic!**

## Final topics

5–10 pages is a good goal for the final paper. (Please don't feel bound to this. Write the best paper you can, which may not be the paper that conforms to a page length guideline.) (At the same time, some of these may be miscalibrated. If something is too hard, that's OK: bite off a chunk you're comfortable with and write about that. If something is too easy, find an extension of it or pick another topic.)

- Intersection theory in the Euclidean plane (cf. page 44, axiom B4), orientations and orientability more generally, the non-orientability of the projective extension $\mathbb{R}\mathrm{P}^2$.
- Nonarchimedean geometry (cf. page 45, axiom A).
- Squaring polygons vs the circle. The Lindemann–Weierstrass theorem: $\pi$ is transcendental.
- Regular polyhedra and Euler characteristics. It's known that there are also only 6 regular polyhedra in $\mathbb{R}^3$ and $\mathbb{R}^4$, and also that all other $\mathbb{R}^n$ have 3 regular polyhedra. This is related to Klein's theorem about finite subgroups of $\mathrm{Isom}(\mathbb{R}^2)$ and its variations. (Stillwell has a Notices article called "The story of the 120-cell" that might be a good read.) (You could even fixate on one particularly complicated solid, like the icosahedron and some of the features of its isometry group.)
- Some small part of differential geometry. Computing the curvature of a space? Computing the curvature of some of the models of hyperbolic geometry discussed? A statement of Gauss's theorema egregium?
- Why do we consider algebraic functions rather than their graphs in positive characteristic? (What's the difference between $x$ and $x^p$ as functions on $\mathbb{F}_p$?)
- Work out the spatial Desargues's implication that $\mathbb{O}\mathrm{P}^3$ does not exist.
- Work out the existence of the exceptional Lie groups. (I forget how this goes, but I have notes: 4/7–4/26. I think you look at symmetry groups preserving certain forms on $\mathbb{O}\mathrm{P}^2$. This may be too advanced, if forms are involved.) (`http://mathoverflow.net/questions/228491/rational-cohomology-of-the-rosenfeld-projective-pl` , `http://math.ucr.edu/home/baez/octonions/node16.html`)
- How do you know when you've inflated your geometry enough to catch "all" the interesting numerical invariants of it? $\mathbb{R}\mathrm{P}^1$ has no concept of length or angle, but the geometry of $\mathbb{R}\mathrm{P}^1$ extends over the half-plane $\mathfrak{h}$ where length and angle do exist. Are there related concepts for the map $\mathbb{F}_p P^2 \to \overline{\mathbb{F}_p} P^2$ or $\mathbb{Q}_p P^2 \to \overline{\mathbb{Q}_p} P^2$?
- Summarize the part of the Grundlagen that explains why *the* theory of the real hyperbolic plane is unique (cf. pp. 209-10).
- The modular function $j$ and the moduli of complex elliptic curves. (Its model as a hyperbolic space, and the hyperbolic symmetries of $j$.)
- Are there polite examples of the last thing mentioned in the book: taking a function on $\mathbb{C}\mathrm{P}^1$, extending it to the "half-space model" in $\mathbb{R}^3$, and studying its symmetries there? (Poincare's article about this be in Stillwell's *Sources of hyperbolic geometry*, of which there is no electronic copy.) (The idea behind this is probably related to the double-coset presentation of automorphic forms. In particular, $SO(3,1)/SO(3)$ gives a model of hyperbolic 3–space, it naturally has a copy of $\mathbb{C}\mathrm{P}^1$ sitting at its boundary, and some accidental isomorphisms will make it line up with the relevant double-coset space for the space of automorphic forms that $j$ belongs to.)
- Various features of geometry on the torus. For instance: intersection properties of lines, its curvature, ....
- Jacob asked a nice question about Desarguesian-style inversion for the action of $PGL_2$ on the projective line at `http://mathoverflow.net/questions/66865/action-of-pgl2-on-projective-space` . Frank Caligari redirected him to the following MathSciNet review:

    MR1020218 (91b:20004b) Reviewed Kerby, William(D-HAMB) A class of canonical sharply 3-transitive groups. Results Math. 16 (1989), no. 1-2, 89–106. 20B22 (20B20)
- Banach–Tarski.
- Use of stereographic projections in "pole figures" and electron diffraction in crystallography.
- Geometries on 3–manifolds and the Perelman's Geometrization theorem.
- Non-Euclidean parabolic geometries.
- Study the intuitionistic version of Desarguesian reconstruction: Desarguesian projective planes in an intuitionistic logic all arise from local rings. (Kryftis's PhD thesis: `http://arxiv.org/abs/1601.04998` .)
- The moduli of quadratics and the moduli of lines. Compactifications of these moduli.
- Foldable numbers and synthetic origami, cubics and the algebraic closure of $\mathbb{Q}$.

- Someone just recently solved sphere packing in two large dimensions! The proof looks quite complicated, so any kind of summary over even a part of it could be interesting. Here's the article: `http://arxiv.org/pdf/1603.04246v1.p` and here's a plainclothes article: `https://www.quantamagazine.org/20160330-sphere-packing-solved-in-higher-di`
- This guy's website is full of interesting ideas: `http://lukyanenko.net/`
- **Your own topic!**