1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

2. $\sum_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5.

In other words, it means that $\lim_{n\to\infty} s_n = 5$, where s_n is the nth partial sum, that is, $\sum_{i=1}^{n} a_i$.

9. (a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent by (11.1.1).

(b) Since $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the *series* $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence.

12. $\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - 1 + \cdots$ is a geometric series with ratio $r = -2$. Since $|r| = 2 > 1$, the series diverges.

15. $\sum_{n=1}^{\infty} 6(0.9)^{n-1}$ is a geometric series with first term $a = 6$ and ratio $r = 0.9$. Since $|r| = 0.9 < 1$, the series converges to

$\frac{a}{1-r} = \frac{6}{1-0.9} = \frac{6}{0.1} = 60.$

22. $\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$ diverges by the Test for Divergence since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{2n-3} = \frac{1}{2} \neq 0.$

26. $\sum_{n=1}^{\infty} \frac{1+3^n}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{3^n}{2^n}\right) = \sum_{n=1}^{\infty} \left[\left(\frac{1}{2}\right)^n + \left(\frac{3}{2}\right)^n\right] = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$. The first series is a convergent geometric series ($|r| = \frac{1}{2} < 1$), but the second series is a divergent geometric series ($|r| = \frac{3}{2} \geq 1$), so the original series is divergent.

35. Using partial fractions, the partial sums of the series $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ are

$$s_n = \sum_{i=2}^{n} \frac{2}{(i-1)(i+1)} = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i+1}\right)$$

$$= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right)$$

This sum is a telescoping series and $s_n = 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}$.

Thus, $\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}\right) = \frac{3}{2}$.

40. For the series $\sum_{n=1}^{\infty}\left(\cos\frac{1}{n^2}-\cos\frac{1}{(n+1)^2}\right)$,

$$s_n=\sum_{i=1}^{n}\left(\cos\frac{1}{i^2}-\cos\frac{1}{(i+1)^2}\right)=\left(\cos 1-\cos\frac{1}{4}\right)+\left(\cos\frac{1}{4}-\cos\frac{1}{9}\right)+\cdots+\left(\cos\frac{1}{n^2}-\cos\frac{1}{(n+1)^2}\right)$$

$$=\cos 1-\cos\frac{1}{(n+1)^2}\quad\text{[telescoping series]}$$

Thus, $\sum_{n=1}^{\infty}\left(\cos\frac{1}{n^2}-\cos\frac{1}{(n+1)^2}\right)=\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(\cos 1-\cos\frac{1}{(n+1)^2}\right)=\cos 1-\cos 0=\cos 1-1.$

Converges

47. $\sum_{n=1}^{\infty}\frac{x^n}{3^n}=\sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n$ is a geometric series with $r=\frac{x}{3}$, so the series converges $\Leftrightarrow\ |r|<1\ \Leftrightarrow\ \frac{|x|}{3}<1\ \Leftrightarrow\ |x|<3$;

that is, $-3<x<3$. In that case, the sum of the series is $\frac{a}{1-r}=\frac{x/3}{1-x/3}=\frac{x/3}{1-x/3}\cdot\frac{3}{3}=\frac{x}{3-x}$.

52. Because $\frac{1}{n}\to 0$ and ln is continuous, we have $\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)=\ln 1=0$.

We now show that the series $\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)=\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)=\sum_{n=1}^{\infty}[\ln(n+1)-\ln n]$ diverges.

$s_n=(\ln 2-\ln 1)+(\ln 3-\ln 2)+\cdots+(\ln(n+1)-\ln n)=\ln(n+1)-\ln 1=\ln(n+1)$.

As $n\to\infty$, $s_n=\ln(n+1)\to\infty$, so the series diverges.

58. (a) Initially, the ball falls a distance H, then rebounds a distance rH, falls rH, rebounds r^2H, falls r^2H, etc. The total distance it travels is

$$H + 2rH + 2r^2H + 2r^3H + \cdots = H(1 + 2r + 2r^2 + 2r^3 + \cdots) = H[1 + 2r(1 + r + r^2 + \cdots)]$$
$$= H\left[1 + 2r\left(\frac{1}{1-r}\right)\right] = H\left(\frac{1+r}{1-r}\right) \text{ meters}$$

(b) From Example 3 in Section 2.1, we know that a ball falls $\frac{1}{2}gt^2$ meters in t seconds, where g is the gravitational acceleration. Thus, a ball falls h meters in $t = \sqrt{2h/g}$ seconds. The total travel time in seconds is

$$\sqrt{\frac{2H}{g}} + 2\sqrt{\frac{2H}{g}r} + 2\sqrt{\frac{2H}{g}r^2} + 2\sqrt{\frac{2H}{g}r^3} + \cdots = \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r} + 2\sqrt{r}^{\,2} + 2\sqrt{r}^{\,3} + \cdots\right]$$
$$= \sqrt{\frac{2H}{g}}\left(1 + 2\sqrt{r}\left[1 + \sqrt{r} + \sqrt{r}^{\,2} + \cdots\right]\right)$$
$$= \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r}\left(\frac{1}{1-\sqrt{r}}\right)\right] = \sqrt{\frac{2H}{g}}\,\frac{1+\sqrt{r}}{1-\sqrt{r}}$$

(c) It will help to make a chart of the time for each descent and each rebound of the ball, together with the velocity just before and just after each bounce. Recall that the time in seconds needed to fall h meters is $\sqrt{2h/g}$. The ball hits the ground with velocity $-g\sqrt{2h/g} = -\sqrt{2hg}$ (taking the upward direction to be positive) and rebounds with velocity $kg\sqrt{2h/g} = k\sqrt{2hg}$, taking time $k\sqrt{2h/g}$ to reach the top of its bounce, where its velocity is 0. At that point, its height is k^2h. All these results follow from the formulas for vertical motion with gravitational acceleration $-g$:
$\frac{d^2y}{dt^2} = -g \quad\Rightarrow\quad v = \frac{dy}{dt} = v_0 - gt \quad\Rightarrow\quad y = y_0 + v_0t - \frac{1}{2}gt^2$.

number of descent	time of descent	speed before bounce	speed after bounce	time of ascent	peak height
1	$\sqrt{2H/g}$	$\sqrt{2Hg}$	$k\sqrt{2Hg}$	$k\sqrt{2H/g}$	k^2H
2	$\sqrt{2k^2H/g}$	$\sqrt{2k^2Hg}$	$k\sqrt{2k^2Hg}$	$k\sqrt{2k^2H/g}$	k^4H
3	$\sqrt{2k^4H/g}$	$\sqrt{2k^4Hg}$	$k\sqrt{2k^4Hg}$	$k\sqrt{2k^4H/g}$	k^6H
...	...	...	...	...	...

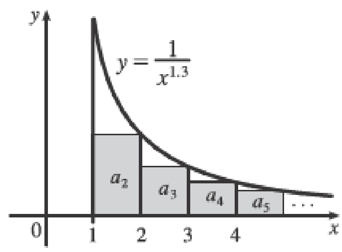
The total travel time in seconds is

$$\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + \cdots = \sqrt{\frac{2H}{g}}(1 + 2k + 2k^2 + 2k^3 + \cdots)$$
$$= \sqrt{\frac{2H}{g}}[1 + 2k(1 + k + k^2 + \cdots)]$$
$$= \sqrt{\frac{2H}{g}}\left[1 + 2k\left(\frac{1}{1-k}\right)\right] = \sqrt{\frac{2H}{g}}\,\frac{1+k}{1-k}$$

Another method: We could use part (b). At the top of the bounce, the height is $k^2h = rh$, so $\sqrt{r} = k$ and the result follows from part (b).

1. The picture shows that $a_2 = \dfrac{1}{2^{1.3}} < \displaystyle\int_1^2 \frac{1}{x^{1.3}}\,dx$,

$a_3 = \dfrac{1}{3^{1.3}} < \displaystyle\int_2^3 \frac{1}{x^{1.3}}\,dx$, and so on, so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}}\,dx$. The integral converges by (7.8.2) with $p = 1.3 > 1$, so the series converges.



4. The function $f(x) = 1/x^5$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{1}{x^5}\,dx = \lim_{t\to\infty} \int_1^t x^{-5}\,dx = \lim_{t\to\infty} \left[\frac{x^{-4}}{-4}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{4t^4} + \frac{1}{4}\right) = \frac{1}{4}.$$

Since this improper integral is convergent, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ is also convergent by the Integral Test.

8. The function $f(x) = \dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^t \left(1 + \frac{1}{x+1}\right) dx = \lim_{t\to\infty} \left[x + \ln(x+1)\right]_1^t = \lim_{t\to\infty} (t + \ln(t+1) - 1 - \ln 2) = \infty, \text{ so}$$

$\displaystyle\int_1^{\infty} \frac{x+2}{x+1}\,dx$ is divergent and the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n+1}$ is divergent.

Note: $\displaystyle\lim_{n\to\infty} \frac{n+2}{n+1} = 1$, so the given series diverges by the Test for Divergence.

10. $\displaystyle\sum_{n=1}^{\infty} n^{-1.4}$ and $\displaystyle\sum_{n=1}^{\infty} n^{-1.2}$ are p-series with $p > 1$, so they converge by (1). Thus, $\displaystyle\sum_{n=1}^{\infty} 3n^{-1.2}$ converges by Theorem 8(i) in Section 11.2. It follows from Theorem 8(ii) that the given series $\displaystyle\sum_{n=1}^{\infty} (n^{-1.4} + 3n^{-1.2})$ also converges.

18. The function $f(x) = \dfrac{3x+2}{x(x+1)} = \dfrac{2}{x} + \dfrac{1}{x+1}$ [by partial fractions] is continuous, positive, and decreasing on $[1, \infty)$ since it is the sum of two such functions. Thus, we can apply the Integral Test.

$$\int_1^{\infty} \frac{3x+2}{x(x+1)}\,dx = \lim_{t\to\infty} \int_1^t \left[\frac{2}{x} + \frac{1}{x+1}\right] dx = \lim_{t\to\infty} \left[2\ln x + \ln(x+1)\right]_1^t = \lim_{t\to\infty} [2\ln t + \ln(t+1) - \ln 2] = \infty.$$

Thus, the series $\displaystyle\sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)}$ diverges.

25. The function $f(x) = \dfrac{1}{x^3 + x}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies. We use partial fractions to evaluate the integral:

$$\int_1^\infty \frac{1}{x^3+x}\,dx = \lim_{t\to\infty}\int_1^t \left[\frac{1}{x} - \frac{x}{1+x^2}\right]dx = \lim_{t\to\infty}\left[\ln x - \frac{1}{2}\ln(1+x^2)\right]_1^t = \lim_{t\to\infty}\left[\ln\frac{x}{\sqrt{1+x^2}}\right]_1^t$$

$$= \lim_{t\to\infty}\left(\ln\frac{t}{\sqrt{1+t^2}} - \ln\frac{1}{\sqrt{2}}\right) = \lim_{t\to\infty}\left(\ln\frac{1}{\sqrt{1+1/t^2}} + \frac{1}{2}\ln 2\right) = \frac{1}{2}\ln 2$$

so the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^3+n}$ converges.

27. We have already shown (in Exercise 21) that when $p = 1$ the series $\sum\limits_{n=2}^{\infty}\dfrac{1}{n(\ln n)^p}$ diverges, so assume that $p \neq 1$.

$f(x) = \dfrac{1}{x(\ln x)^p}$ is continuous and positive on $[2, \infty)$, and $f'(x) = -\dfrac{p + \ln x}{x^2(\ln x)^{p+1}} < 0$ if $x > e^{-p}$, so that f is eventually decreasing and we can use the Integral Test.

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^{1-p}}{1-p}\right]_2^t \quad [\text{for } p \neq 1] \;=\; \lim_{t\to\infty}\left[\frac{(\ln t)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p}\right]$$

This limit exists whenever $1 - p < 0 \quad\Leftrightarrow\quad p > 1$, so the series converges for $p > 1$.

29. Clearly the series cannot converge if $p \geq -\frac{1}{2}$, because then $\lim\limits_{n\to\infty} n(1+n^2)^p \neq 0$. So assume $p < -\frac{1}{2}$. Then

$f(x) = x(1+x^2)^p$ is continuous, positive, and eventually decreasing on $[1, \infty)$, and we can use the Integral Test.

$$\int_1^\infty x(1+x^2)^p\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{(1+x^2)^{p+1}}{p+1}\right]_1^t = \frac{1}{2(p+1)}\lim_{t\to\infty}\left[(1+t^2)^{p+1} - 2^{p+1}\right].$$

This limit exists and is finite $\quad\Leftrightarrow\quad p+1<0 \quad\Leftrightarrow\quad p<-1$, so the series converges whenever $p < -1$.