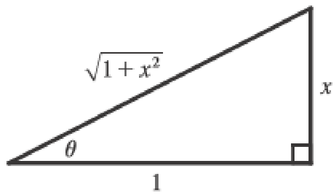
26. $\displaystyle\int_0^\infty \frac{x\arctan x}{(1+x^2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x\arctan x}{(1+x^2)^2}\,dx$. Let $u=\arctan x$, $dv = \dfrac{x\,dx}{(1+x^2)^2}$. Then $du = \dfrac{dx}{1+x^2}$,

$v = \dfrac{1}{2}\displaystyle\int \frac{2x\,dx}{(1+x^2)^2} = \frac{-1/2}{1+x^2}$, and

$$\begin{aligned}\int \frac{x\arctan x}{(1+x^2)^2}\,dx &= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{2}\int\frac{dx}{(1+x^2)^2} \quad \left[\begin{array}{l} x=\tan\theta, \\ dx = \sec^2\theta\,d\theta\end{array}\right] \\ &= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{2}\int\frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^2} \\ &= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{2}\int\cos^2\theta\,d\theta \\ &= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{\theta}{4} + \frac{\sin\theta\,\cos\theta}{4} + C \\ &= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{4}\arctan x + \frac{1}{4}\frac{x}{1+x^2} + C\end{aligned}$$



It follows that

$$\begin{aligned}\int_0^\infty \frac{x\arctan x}{(1+x^2)^2}\,dx &= \lim_{t\to\infty}\left[-\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{4}\arctan x + \frac{1}{4}\frac{x}{1+x^2}\right]_0^t \\ &= \lim_{t\to\infty}\left(-\frac{1}{2}\frac{\arctan t}{1+t^2} + \frac{1}{4}\arctan t + \frac{1}{4}\frac{t}{1+t^2}\right) = 0 + \frac{1}{4}\cdot\frac{\pi}{2} + 0 = \frac{\pi}{8}.\end{aligned}$$

Convergent

32. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\left[\sin^{-1}x\right]_0^t = \lim_{t\to 1^-}\sin^{-1}t = \frac{\pi}{2}.$ Convergent

39. $I = \int_0^2 z^2\ln z\,dz = \displaystyle\lim_{t\to 0^+}\int_t^2 z^2\ln z\,dz = \lim_{t\to 0^+}\left[\frac{z^3}{3^2}(3\ln z - 1)\right]_t^2$ $\left[\begin{array}{l}\text{integrate by parts} \\ \text{or use Formula 101}\end{array}\right]$

$= \displaystyle\lim_{t\to 0^+}\left[\tfrac{8}{9}(3\ln 2 - 1) - \tfrac{1}{9}t^3(3\ln t - 1)\right] = \tfrac{8}{3}\ln 2 - \tfrac{8}{9} - \tfrac{1}{9}\lim_{t\to 0^+}\left[t^3(3\ln t - 1)\right] = \tfrac{8}{3}\ln 2 - \tfrac{8}{9} - \tfrac{1}{9}L.$

Now $L = \displaystyle\lim_{t\to 0^+}\left[t^3(3\ln t - 1)\right] = \lim_{t\to 0^+}\frac{3\ln t - 1}{t^{-3}} \overset{\text{H}}{=} \lim_{t\to 0^+}\frac{3/t}{-3/t^4} = \lim_{t\to 0^+}\left(-t^3\right) = 0.$

Thus, $L = 0$ and $I = \frac{8}{3}\ln 2 - \frac{8}{9}$. Convergent

55. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \int_0^1 \frac{dx}{\sqrt{x}\,(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{\sqrt{x}\,(1+x)} + \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}\,(1+x)}$. Now

$\displaystyle\int \frac{dx}{\sqrt{x}\,(1+x)} = \int\frac{2u\,du}{u(1+u^2)}$ $\left[\begin{array}{c} u=\sqrt{x}, x = u^2, \\ dx = 2u\,du\end{array}\right]$ $= 2\displaystyle\int\frac{du}{1+u^2} = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C$, so

$$\begin{aligned}\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} &= \lim_{t\to 0^+}\left[2\tan^{-1}\sqrt{x}\right]_t^1 + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^t \\ &= \lim_{t\to 0^+}\left[2\left(\tfrac{\pi}{4}\right) - 2\tan^{-1}\sqrt{t}\right] + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{t} - 2\left(\tfrac{\pi}{4}\right)\right] = \tfrac{\pi}{2} - 0 + 2\left(\tfrac{\pi}{2}\right) - \tfrac{\pi}{2} = \pi.\end{aligned}$$

60. (a) $\boldsymbol{n=0}$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t = \lim_{t\to\infty}\left[-e^{-t}+1\right] = 0+1=1$

$\boldsymbol{n=1}$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx$. To evaluate $\displaystyle\int xe^{-x}\,dx$, we'll use integration by parts with $u=x$, $dv=e^{-x}\,dx \quad\Rightarrow\quad du=dx$, $v=-e^{-x}$.

So $\displaystyle\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} - e^{-x} + C = (-x-1)e^{-x}+C$ and

$$\lim_{t\to\infty}\int_0^t xe^{-x}\,dx = \lim_{t\to\infty}\left[(-x-1)e^{-x}\right]_0^t = \lim_{t\to\infty}\left[(-t-1)e^{-t}+1\right] = \lim_{t\to\infty}\left[-te^{-t}-e^{-t}+1\right]$$

$$= 0-0+1 \quad [\text{use l'Hospital's Rule}] \quad = 1$$

$\boldsymbol{n=2}$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx$. To evaluate $\displaystyle\int x^2e^{-x}\,dx$, we could use integration by parts again or Formula 97. Thus,

$$\lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx = \lim_{t\to\infty}\left[-x^2e^{-x}\right]_0^t + 2\lim_{t\to\infty}\int_0^t xe^{-x}\,dx$$

$$= 0+0+2(1) \quad [\text{use l'Hospital's Rule and the result for } n=1] \quad = 2$$

$\boldsymbol{n=3}$: $$\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t x^3e^{-x}\,dx \overset{97}{=} \lim_{t\to\infty}\left[-x^3e^{-x}\right]_0^t + 3\lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx$$

$$= 0+0+3(2) \quad [\text{use l'Hospital's Rule and the result for } n=2] \quad = 6$$

(b) For $n=1, 2$, and 3, we have $\int_0^\infty x^n e^{-x}\,dx = 1, 2$, and 6. The values for the integral are equal to the factorials for n, so we guess $\int_0^\infty x^n e^{-x}\,dx = n!$.

(c) Suppose that $\int_0^\infty x^k e^{-x}\,dx = k!$ for some positive integer k. Then $\int_0^\infty x^{k+1}e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}\,dx$.

To evaluate $\int x^{k+1}e^{-x}\,dx$, we use parts with $u=x^{k+1}$, $dv=e^{-x}\,dx \quad\Rightarrow\quad du=(k+1)x^k\,dx$, $v=-e^{-x}$.

So $\int x^{k+1}e^{-x}dx = -x^{k+1}e^{-x} - \int -(k+1)x^ke^{-x}\,dx = -x^{k+1}e^{-x} + (k+1)\int x^ke^{-x}\,dx$ and

$$\lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}\,dx = \lim_{t\to\infty}\left[-x^{k+1}e^{-x}\right]_0^t + (k+1)\lim_{t\to\infty}\int_0^t x^ke^{-x}\,dx$$

$$= \lim_{t\to\infty}\left[-t^{k+1}e^{-t}+0\right] + (k+1)k! = 0+0+(k+1)! = (k+1)!,$$

so the formula holds for $k+1$. By induction, the formula holds for all positive integers. (Since $0!=1$, the formula holds for $n=0$, too.)

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.

(b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.

(c) The terms a_n become large as n becomes large. In fact, we can make a_n as large as we like by taking n sufficiently large.

5. $a_n = \frac{3(-1)^n}{n!}$, so the sequence is $\left\{\frac{-3}{1}, \frac{3}{2}, \frac{-3}{6}, \frac{3}{24}, \frac{-3}{120}, \ldots\right\} = \left\{-3, \frac{3}{2}, -\frac{1}{2}, \frac{1}{8}, -\frac{1}{40}, \ldots\right\}$.

15. The first six terms of $a_n = \frac{n}{2n+1}$ are $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}$. It appears that the sequence is approaching $\frac{1}{2}$.

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+1/n} = \frac{1}{2}$$

16. $\{\cos(n\pi/3)\}_{n=1}^{9} = \left\{\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, -1\right\}$. The sequence does not appear to have a limit. The values will cycle through the first six numbers in the sequence—never approaching a particular number.

18. $a_n = \frac{n^3}{n^3+1} = \frac{n^3/n^3}{(n^3+1)/n^3} = \frac{1}{1+1/n^3}$, so $a_n \to \frac{1}{1+0} = 1$ as $n \to \infty$. Converges

29. $a_n = \frac{(2n-1)!}{(2n+1)!} = \frac{(2n-1)!}{(2n+1)(2n)(2n-1)!} = \frac{1}{(2n+1)(2n)} \to 0$ as $n \to \infty$. Converges

33. $a_n = n^2 e^{-n} = \frac{n^2}{e^n}$. Since $\lim_{x\to\infty} \frac{x^2}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$, it follows from Theorem 3 that $\lim_{n\to\infty} a_n = 0$. Converges

68. (a) Let P_n be the statement that $a_{n+1} \geq a_n$ and $a_n \leq 3$. P_1 is obviously true. We will assume that P_n is true and then show that as a consequence P_{n+1} must also be true. $a_{n+2} \geq a_{n+1} \Leftrightarrow \sqrt{2+a_{n+1}} \geq \sqrt{2+a_n} \Leftrightarrow 2+a_{n+1} \geq 2+a_n \Leftrightarrow a_{n+1} \geq a_n$, which is the induction hypothesis. $a_{n+1} \leq 3 \Leftrightarrow \sqrt{2+a_n} \leq 3 \Leftrightarrow 2+a_n \leq 9 \Leftrightarrow a_n \leq 7$, which is certainly true because we are assuming that $a_n \leq 3$. So P_n is true for all n, and so $a_1 \leq a_n \leq 3$ (showing that the sequence is bounded), and hence by the Monotonic Sequence Theorem, $\lim_{n\to\infty} a_n$ exists.

(b) If $L = \lim_{n\to\infty} a_n$, then $\lim_{n\to\infty} a_{n+1} = L$ also, so $L = \sqrt{2+L} \Rightarrow L^2 = 2+L \Leftrightarrow L^2 - L - 2 = 0 \Leftrightarrow (L+1)(L-2) = 0 \Leftrightarrow L = 2$ [since L can't be negative].