1. (a) $\dfrac{2x}{(x+3)(3x+1)} = \dfrac{A}{x+3} + \dfrac{B}{3x+1}$

(b) $\dfrac{1}{x^3+2x^2+x} = \dfrac{1}{x(x^2+2x+1)} = \dfrac{1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

4. (a) $\dfrac{x^3}{x^2+4x+3} = x - 4 + \dfrac{13x+12}{x^2+4x+3} = x - 4 + \dfrac{13x+12}{(x+1)(x+3)} = x - 4 + \dfrac{A}{x+1} + \dfrac{B}{x+3}$

(b) $\dfrac{2x+1}{(x+1)^3(x^2+4)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3} + \dfrac{Dx+E}{x^2+4} + \dfrac{Fx+G}{(x^2+4)^2}$

8. $\displaystyle\int \frac{r^2}{r+4}\,dr = \int\left(\frac{r^2-16}{r+4} + \frac{16}{r+4}\right)dr = \int\left(r - 4 + \frac{16}{r+4}\right)dr$ [or use long division]

$= \frac{1}{2}r^2 - 4r + 16\ln|r+4| + C$

15. $\dfrac{x^3-2x^2-4}{x^3-2x^2} = 1 + \dfrac{-4}{x^2(x-2)}$. Write $\dfrac{-4}{x^2(x-2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2}$. Multiplying both sides by $x^2(x-2)$ gives

$-4 = Ax(x-2) + B(x-2) + Cx^2$. Substituting 0 for x gives $-4 = -2B \Leftrightarrow B = 2$. Substituting 2 for x gives

$-4 = 4C \Leftrightarrow C = -1$. Equating coefficients of x^2, we get $0 = A + C$, so $A = 1$. Thus,

$$\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx = \int_3^4\left(1 + \frac{1}{x} + \frac{2}{x^2} - \frac{1}{x-2}\right)dx = \left[x + \ln|x| - \frac{2}{x} - \ln|x-2|\right]_3^4$$

$$= \left[\left(4 + \ln 4 - \tfrac{1}{2} - \ln 2\right) - \left(3 + \ln 3 - \tfrac{2}{3} - 0\right)\right] = \tfrac{7}{6} + \ln\tfrac{2}{3}$$

16. $\dfrac{x^3-4x-10}{x^2-x-6} = x + 1 + \dfrac{3x-4}{(x-3)(x+2)}$. Write $\dfrac{3x-4}{(x-3)(x+2)} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$. Then

$3x - 4 = A(x+2) + B(x-3)$. Taking $x = 3$ and $x = -2$, we get $5 = 5A \Leftrightarrow A = 1$ and $-10 = -5B \Leftrightarrow B = 2$, so

$$\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx = \int_0^1\left(x + 1 + \frac{1}{x-3} + \frac{2}{x+2}\right)dx = \left[\frac{1}{2}x^2 + x + \ln|x-3| + 2\ln(x+2)\right]_0^1$$

$$= \left(\tfrac{1}{2} + 1 + \ln 2 + 2\ln 3\right) - (0 + 0 + \ln 3 + 2\ln 2) = \tfrac{3}{2} + \ln 3 - \ln 2 = \tfrac{3}{2} + \ln\tfrac{3}{2}$$

23. $\dfrac{5x^2+3x-2}{x^3+2x^2} = \dfrac{5x^2+3x-2}{x^2(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}$. Multiply by $x^2(x+2)$ to

get $5x^2 + 3x - 2 = Ax(x+2) + B(x+2) + Cx^2$. Set $x = -2$ to get $C = 3$, and take

$x = 0$ to get $B = -1$. Equating the coefficients of x^2 gives $5 = A + C \Rightarrow A = 2$. So

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int\left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right)dx = 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C.$$

42. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2\,du \Rightarrow$

$$\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx = \int_0^1 \frac{3u^2\,du}{1+u} = \int_0^1 \left(3u - 3 + \frac{3}{1+u}\right) du = \left[\tfrac{3}{2}u^2 - 3u + 3\ln(1+u)\right]_0^1 = 3\left(\ln 2 - \tfrac{1}{2}\right).$$

46. Let $u = \sqrt{1+\sqrt{x}}$, so that $u^2 = 1 + \sqrt{x}$, $x = (u^2-1)^2$, and $dx = 2(u^2-1)\cdot 2u\,du = 4u(u^2-1)\,du$. Then

$$\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx = \int \frac{u}{(u^2-1)^2}\cdot 4u(u^2-1)\,du = \int \frac{4u^2}{u^2-1}\,du = \int \left(4 + \frac{4}{u^2-1}\right) du.$$ Now

$\frac{4}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 4 = A(u-1) + B(u+1)$. Setting $u = 1$ gives $4 = 2B$, so $B = 2$. Setting $u = -1$ gives $4 = -2A$, so $A = -2$. Thus,

$$\int \left(4 + \frac{4}{u^2-1}\right) du = \int \left(4 - \frac{2}{u+1} + \frac{2}{u-1}\right) du = 4u - 2\ln|u+1| + 2\ln|u-1| + C$$
$$= 4\sqrt{1+\sqrt{x}} - 2\ln\left(\sqrt{1+\sqrt{x}}+1\right) + 2\ln\left(\sqrt{1+\sqrt{x}}-1\right) + C$$

48. Let $u = \sin x$. Then $du = \cos x\,dx \Rightarrow$

$$\int \frac{\cos x\,dx}{\sin^2 x + \sin x} = \int \frac{du}{u^2+u} = \int \frac{du}{u(u+1)} = \int \left[\frac{1}{u} - \frac{1}{u+1}\right] du = \ln\left|\frac{u}{u+1}\right| + C = \ln\left|\frac{\sin x}{1+\sin x}\right| + C.$$

1. Let $u = \sin x$, so that $du = \cos x\,dx$. Then $\int \cos x(1+\sin^2 x)\,dx = \int (1+u^2)\,du = u + \frac{1}{3}u^3 + C = \sin x + \frac{1}{3}\sin^3 x + C$.

3. $$\int \frac{\sin x + \sec x}{\tan x}\,dx = \int \left(\frac{\sin x}{\tan x} + \frac{\sec x}{\tan x}\right) dx = \int (\cos x + \csc x)\,dx = \sin x + \ln|\csc x - \cot x| + C$$

7. Let $u = \arctan y$. Then $du = \frac{dy}{1+y^2} \Rightarrow \int_{-1}^1 \frac{e^{\arctan y}}{1+y^2}\,dy = \int_{-\pi/4}^{\pi/4} e^u\,du = \left[e^u\right]_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4}$.

14. Let $u = 1 + x^2$, so that $du = 2x\,dx$. Then

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \frac{x^2}{\sqrt{1+x^2}}\,(x\,dx) = \int \frac{u-1}{u^{1/2}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right) du = \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) + C$$
$$= \tfrac{1}{3}(1+x^2)^{3/2} - (1+x^2)^{1/2} + C \quad \left[\text{or } \tfrac{1}{3}(x^2-2)\sqrt{1+x^2} + C\right]$$

18. Let $u = e^{2t}$, $du = 2e^{2t}\,dt$. Then $\int \frac{e^{2t}}{1+e^{4t}}\,dt = \int \frac{\frac{1}{2}(2e^{2t})\,dt}{1+(e^{2t})^2} = \int \frac{\frac{1}{2}\,du}{1+u^2} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1}(e^{2t}) + C$.

22. Let $u = 1 + (\ln x)^2$, so that $du = \frac{2\ln x}{x}\,dx$. Then

$$\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\left(2\sqrt{u}\right) + C = \sqrt{1+(\ln x)^2} + C.$$

41. Let $u = \theta$, $dv = \tan^2\theta\, d\theta = \left(\sec^2\theta - 1\right) d\theta \quad \Rightarrow \quad du = d\theta$ and $v = \tan\theta - \theta$. So

$$\begin{aligned}\int \theta \tan^2\theta\, d\theta &= \theta(\tan\theta - \theta) - \int(\tan\theta - \theta)\, d\theta = \theta\tan\theta - \theta^2 - \ln|\sec\theta| + \tfrac{1}{2}\theta^2 + C \\ &= \theta\tan\theta - \tfrac{1}{2}\theta^2 - \ln|\sec\theta| + C\end{aligned}$$