Math 231 Homework: Trigonometric Integrals and Trigonometric Substitutions

Due Tuesday February 2 at the start of section

Trigonometric Integrals **Section 7.2** # 1,6,12,13,19,23,31,43

7.2.1

$$\int \sin^3(x)\cos^2(x) =$$
$$\int \sin(x)\left(1-\cos^2(x)\right)\cos^2(x)dx = \qquad u=\cos(x) \qquad du = \sin(x)dx$$
$$\int \left(1-u^2\right)u^2du = \int u^2 - u^4du = \frac{u^3}{3} - \frac{u^5}{5} + c =$$
$$\frac{\cos^3(x)}{3} - \frac{\cos^5(x)}{5} + c$$

7.2.6 This requires a u-substitution and an integration by parts.

$$\begin{aligned}
I &= \int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}dx \qquad u=\sqrt{x} \qquad du = \frac{dx}{2\sqrt{x}}\\
&= 2\int \sin^3(u)du = 2\int \sin(u)(1-\cos^2(u))du \qquad v=\cos(u) \qquad dv = -\sin(u)du\\
&= -2\int (1-v^2)dv = \frac{2v^3}{3} - 2v + c\\
&= \frac{2\cos^3(\sqrt{x})}{3} - 2\cos(\sqrt{x}) + c
\end{aligned}$$

7.2.12 This requires a trig identity plus an integration by parts.

$$\begin{aligned}
I &= \int x\cos^2(x)dx = \int \frac{x}{2}\left(1+\cos(2x)\right)dx\\
&= \frac{x^2}{4} + \int \frac{x}{2}\cos(2x)dx \qquad u = x/2 \quad dv = \cos(2x)dx \qquad du = dx/2 \quad v = \frac{\sin(2x)}{2}\\
&= \frac{x^2}{4} + \frac{x\sin(2x)}{4} - \frac{1}{4}\int \sin(2x)dx\\
&= \frac{x^2}{4} + \frac{x\sin(2x)}{4} + \frac{1}{8}\cos(2x) + c
\end{aligned}$$

7.2.13 This is easiest to do with a trig identity. There are **LOTS** of different ways to do this. The answers will all look a little different but will be related by trig identities

$$\begin{aligned}
I &= \int \sin^2(x)\cos^2(x)dx \qquad 2\sin(x)\cos(x) = \sin(2x)\\
&= \frac{1}{4}\int \sin^2(2x)dx \qquad \sin^2(2x) = \frac{1-cos(4x)}{2}\\
&= \frac{1}{8}\int 1-\cos(4x)dx = \frac{x}{8} - \frac{\sin(4x)}{32} + c
\end{aligned}$$

7.2.19 More trig identities

$$\begin{aligned} I &= \int \frac{\cos(x) + \sin(2x)}{\sin(x)} \\ &= \int \frac{\cos(x) + 2\sin(x)\cos(x)}{\sin(x)} \\ &= \int \frac{\cos(x)}{\sin(x)} dx + 2\int \cos(x) dx \\ &= \ln|\sin(x)| + 2\sin(x) + c \end{aligned}$$

where the integral of $\cot(x)$ can be done by a u-sub if you didn't know it offhand.

7.2.23 More identities to the rescue:

$$\begin{aligned} I &= \int \tan^2(x) dx = \int \frac{\sin^2(x)}{\cos^2(x)} dx \\ &= \int \frac{1 - \cos^2(x)}{\cos^2(x)} dx \\ &= \int \sec^2(x) - 1 dx = \tan(x) - x + c \end{aligned}$$

7.2.31 Remember the rule: If $\tan(x)$ appears to an odd power you can make the change of variable $u = \sec(x)$ and get the integral of powers of u

$$\begin{aligned} I &= \int \tan^5(x) dx \\ &= \int \tan(x)(tan^2(x))^2 dx \\ &= \int \tan(x)(\sec^2(x) - 1)^2 dx \qquad u = \sec(x) \qquad du = \sec(x)\tan(x) dx \\ &= \int \frac{(u^2-1)^2}{u} du = \int (u^3 - 2u + \frac{1}{u}) du \\ &= \frac{u^4}{4} - u^2 + \ln|u| + c \\ &= \frac{\sec^4(x)}{4} - \sec^2(x) + \ln|\sec(x)| + c \end{aligned}$$

7.2.43 I suggest doing these by integrating by parts twice. Doing so gives

$$\begin{aligned} I &= \sin(8x)\cos(5x) dx \qquad u = \sin(8x) \quad dv = \cos(5x) dx \qquad v = \frac{\sin(5x)}{5} \quad du = 8\cos(8x) dx \\ &= \frac{\sin(8x)\sin(5x)}{5} - \frac{8}{5}\int \cos(8x)\sin(5x) dx \qquad u = \cos(8x) \quad dv = \sin(5x) dx \qquad v = -\frac{\cos(5x)}{5} \quad du = \cdot \\ &= \frac{\sin(8x)\sin(5x)}{5} + \frac{8}{25}\cos(5x)\cos(8x) - \frac{64}{25}\int \sin(8x)\cos(5x) dx \\ I &= \frac{\sin(8x)\sin(5x)}{5} + \frac{8}{25}\cos(5x)\cos(8x) - \frac{64}{25} I \\ \frac{89}{25} I &= \frac{\sin(8x)\sin(5x)}{5} + \frac{8}{25}\cos(5x)\cos(8x) \\ I &= \frac{5}{89}\sin(8x)\sin(5x) + \frac{8}{89}\cos(5x)\cos(8x) + c \end{aligned}$$

You can also use the identity

$$\sin(a)\cos(b) = \frac{\sin(a+b)+\sin(a-b)}{2}$$

which will give an answer which looks completely different but is actually the same.

Section 7.3 # 1,3,7,11,19,22,33,43

7.3.1

$$\begin{aligned} I &= \int \frac{dx}{x^2\sqrt{x^2-9}} \qquad x = 3\sec(u) \quad dx = 3\sec(u)\tan(u)du \\ &= \int \frac{3\sec(u)\tan(u)du}{9\sec^2(u)\sqrt{9\sec^2(u)-9}} \\ &= \int \frac{3\sec(u)\tan(u)du}{27\sec^2(u)\tan(u)} \\ &= \int \frac{1}{9\sec(u)} = \frac{1}{9}\int \cos(u)du = \frac{1}{9}\sin(u) + c \end{aligned}$$

Drawing the triangle (done in lecture) gives

$$\sin(u) = \frac{\sqrt{x^2-9}}{x}$$

7.3.3

$$\begin{aligned} I &= \int \frac{x^3}{\sqrt{x^2+9}} \qquad x = 3\tan(u) \quad dx = 3\sec^2(u)du \\ &= \int \frac{27\tan^3(u)}{3\sec(u)} 3\sec^2(u)du \\ &= \int 27\tan^3(u)\sec(u)du \end{aligned}$$

Here we make the substitution $v = \sec(u) \quad dv = \sec(u)\tan(u)du$ to give

$$\begin{aligned} I &= 27\int \tan^2(u)\tan(u)\sec(u)du \qquad v = \sec(u) \quad dv = \sec(u)\tan(u)du \\ &= 27\int (v^2-1)dv = \frac{v^3}{3} - v + c = \frac{\sec^3(u)}{3} - \sec(u) + c \\ &= 9(1+\frac{x^2}{9})^{\frac{3}{2}} - 27\sqrt{1+\frac{x^2}{9}} + c \end{aligned}$$

Again we used the triangle to find $\sec(u)$ in terms of $x/3 = \tan(u)$

7.3.7

$$\begin{aligned} I &= \int \frac{1}{x^2\sqrt{25-x^2}} \qquad x = 5\sin(u) \quad dx = 5\cos(u)du \\ &= \int \frac{1}{125\sin^2(u)\cos(u)} 5\cos(u)du \\ &= \int \frac{1}{25}\csc^2(u)du \\ &= -\cot(u) + c = -\frac{\sqrt{25-x^2}}{x} + c \end{aligned}$$

7.3.11

$$\begin{aligned} I &= \int \sqrt{1-4x^2}dx \qquad x = \frac{\sin(u)}{2} \quad dx = \frac{\cos(u)}{2}du \\ &= \int \cos(u)\frac{\cos(u)}{2}du \\ &= \int \frac{1}{4}\left(1+\cos(2u)\right)du \\ &= \frac{u}{4} + \frac{1}{8}\sin(2u) + c \end{aligned}$$

To put this back in terms of x note that $\sin(2u) = 2\sin(u)\cos(u)$ and that $u = \arcsin(2x)$. This gives

$$I = \frac{1}{4}\arcsin(2x) + \frac{1}{2}x\sqrt{1-4x^2} + c$$

7.3.19

This requires a little bit of cleverness to spot.....

$$\begin{aligned} I &= \int \frac{\sqrt{1+x^2}}{x}dx \qquad x = \tan(u) \quad dx = \sec^2(u)du \\ &= \int \frac{\sqrt{1+\tan^2(u)}}{\tan(u)}\sec^2(u)du \\ &= \int \frac{\sqrt{\sec^2(u)}}{\tan(u)}\sec^2(u)du \\ &= \int \frac{\sec^3(u)}{\tan(u)}du \\ &= \int \frac{\sec(u)}{\tan(u)}(1+\tan^2(u))du \\ &= \int \csc(u) + \sec(u)\tan(u) \\ &= \ln|\csc(u) - \cot(u)| + \sec(u) + c \\ &= \ln|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}| + \sqrt{1+x^2} + c \end{aligned}$$

7.3.22

$$\begin{aligned} I &= \int \sqrt{x^2+1}dx \qquad x = \tan(u) \quad dx = \sec^2(u)du \\ &= \int \sec^3(u)du \end{aligned}$$

The integral $\int \sec^3(x)dx$ is done in the book. It is given by

$$\int \sec^3(u)du = \frac{1}{2}\left(\sec(u)\tan(u) + \ln|\sec(u)+\tan(u)|\right)$$

This gives

$$I = \frac{1}{2}\left(\sec(u)\tan(u) + \ln|\sec(u)+\tan(u)|\right)|_0^{\frac{\pi}{4}} = \frac{1}{2}\left(\sqrt{2} + \ln|1+\sqrt{2}|\right)$$

7.3.33 The mean is by definition

$$\begin{aligned} M &= \frac{1}{6}\int_1^7 \frac{\sqrt{x^2-1}}{x}dx \qquad x=\sec(u) \quad dx=\sec(u)\tan(u)du \\ &= \frac{1}{6}\int_0^\alpha \frac{\tan(u)}{\sec(u)}\sec(u)\tan(u)du \\ &= \frac{1}{6}\int_0^\alpha \tan^2(u)du \\ &= \frac{1}{6}\int_0^\alpha (\sec^2(u)-1)du \\ &= \frac{1}{6}(\tan(u)-u)|_0^\alpha \end{aligned}$$

where $\sec(\alpha)=7$. Drawing the triangle shows that it is a $(1,\sqrt{48},7)$ right triangle so this is

$$M=\frac{1}{6}\left(\sqrt{48}-\alpha\right)$$

where $\alpha=\text{arcsec}(7)=\arcsin(\frac{\sqrt{48}}{7})$.

7.3.43 Compute the volume of the torus (doughnut) formed by rotating the circle

$$x^2+(y-R)^2=r^2$$

about the x-axis.

The volume is $2\pi^2Rr^2$. If you set this up correctly the integral is pretty easy to do. I used the method of washers. If you draw a little picture it is easy to see that if you fill the doughnut with washers in the $z=\textit{constant}$ plane then they have inner radius $R-\sqrt{r^2-z^2}$ and outer radius $R+\sqrt{r^2-z^2}$ The area of one such washer is

$$\begin{aligned} A &= \pi\left((R+\sqrt{r^2-z^2})^2-(R-\sqrt{r^2-z^2})\right) \\ &= 4\pi R\sqrt{r^2-z^2} \end{aligned}$$

Now we have to integrate this over $z\in(-r,r)$ to get the volume of the doughnut. This is

$$\begin{aligned} V &= \int A(z)dz \\ &= \int_{-r}^{r} 4\pi R\sqrt{r^2-z^2}dz \end{aligned}$$

If you're a clever bastard you can just notice that $\int_{-r}^{r}\sqrt{r^2-z^2}dz$ is the volume of a semi-circle of radius r and thus is $\frac{\pi r^2}{2}$. This gives

$$V=4\pi R\frac{\pi r^2}{2}=2\pi^2Rr^2$$

However since this is the chapter on trig substitutions you're probably expected to make one. The substitution $z=r\sin(u)$ converts this to

$$\begin{aligned} V &= 4\pi Rr^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2(u)du \\ &= 4\pi Rr^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2}\left(1-\cos(2u)\right)du \\ &= 2\pi Rr^2(u-\cos(2u))|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=2\pi^2Rr^2 \end{aligned}$$

You could evaluate it with cylindrical shells instead of washers, and it probably gives a similar looking integral, but I didn't do this.