4. If $a_n = \dfrac{(-1)^n x^n}{n+1}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{n+2}\cdot\dfrac{n+1}{x^n}\right| = \lim\limits_{n\to\infty}\dfrac{|x|}{1+1/(n+1)} = |x|$.

By the Ratio Test, the series $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n x^n}{n+1}$ converges when $|x| < 1$, so $R = 1$. When $x = -1$, the series diverges because it is the harmonic series; when $x = 1$, it is the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I = (-1, 1]$.

10. If $a_n = \dfrac{10^n x^n}{n^3}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{10^{n+1}x^{n+1}}{(n+1)^3}\cdot\frac{n^3}{10^n x^n}\right| = \lim_{n\to\infty}\left|\frac{10x\,n^3}{(n+1)^3}\right| = \lim_{n\to\infty}\frac{10\,|x|}{(1+1/n)^3} = \frac{10\,|x|}{1^3} = 10\,|x|$$

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{10^n x^n}{n^3}$ converges when $10\,|x| < 1 \quad\Leftrightarrow\quad |x| < \frac{1}{10}$, so the radius of convergence is $R = \frac{1}{10}$. When $x = -\frac{1}{10}$, the series converges by the Alternating Series Test; when $x = \frac{1}{10}$, the series converges because it is a p-series with $p = 3 > 1$. Thus, the interval of convergence is $I = \left[-\frac{1}{10}, \frac{1}{10}\right]$.

13. If $a_n = (-1)^n\dfrac{x^n}{4^n \ln n}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{4^{n+1}\ln(n+1)}\cdot\dfrac{4^n \ln n}{x^n}\right| = \dfrac{|x|}{4}\lim\limits_{n\to\infty}\dfrac{\ln n}{\ln(n+1)} = \dfrac{|x|}{4}\cdot 1$

[by l'Hospital's Rule] $= \dfrac{|x|}{4}$. By the Ratio Test, the series converges when $\dfrac{|x|}{4} < 1 \quad\Leftrightarrow\quad |x| < 4$, so $R = 4$. When $x = -4$, $\sum\limits_{n=2}^{\infty}(-1)^n\dfrac{x^n}{4^n \ln n} = \sum\limits_{n=2}^{\infty}\dfrac{[(-1)(-4)]^n}{4^n \ln n} = \sum\limits_{n=2}^{\infty}\dfrac{1}{\ln n}$. Since $\ln n < n$ for $n \ge 2$, $\dfrac{1}{\ln n} > \dfrac{1}{n}$ and $\sum\limits_{n=2}^{\infty}\dfrac{1}{n}$ is the divergent harmonic series (without the $n = 1$ term), $\sum\limits_{n=2}^{\infty}\dfrac{1}{\ln n}$ is divergent by the Comparison Test. When $x = 4$, $\sum\limits_{n=2}^{\infty}(-1)^n\dfrac{x^n}{4^n \ln n} = \sum\limits_{n=2}^{\infty}(-1)^n\dfrac{1}{\ln n}$, which converges by the Alternating Series Test. Thus, $I = (-4, 4]$.

14. $a_n = (-1)^n\dfrac{x^{2n}}{(2n)!}$, so $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{|x|^{2n+2}}{(2n+2)!}\cdot\dfrac{(2n)!}{|x|^{2n}} = \lim\limits_{n\to\infty}\dfrac{|x|^2}{(2n+1)(2n+2)} = 0 < 1$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

28. If $a_n = \dfrac{n!\,x^n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)}\cdot\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,x^n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x|}{2n+1} = \tfrac{1}{2}\,|x|.$$

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} a_n$ converges when $\frac{1}{2}\,|x| < 1 \quad\Rightarrow\quad |x| < 2$, so $R = 2$. When $x = \pm 2$,

$$|a_n| = \frac{n!\,2^n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)} = \frac{[1\cdot 2\cdot 3\cdot\cdots\cdot n]\,2^n}{[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)]} = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)} > 1,$$ so both endpoint series diverge by the Test for Divergence. Thus, the interval of convergence is $I = (-2, 2)$.

2. If $f(x) = \sum_{n=0}^{\infty} b_n x^n$ converges on $(-2, 2)$, then $\int f(x)dx = C + \sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$ has the same radius of convergence (by Theorem 2), but may not have the same interval of convergence—it may happen that the integrated series converges at an endpoint (or both endpoints).

7. $f(x) = \frac{x}{9+x^2} = \frac{x}{9}\left[\frac{1}{1+(x/3)^2}\right] = \frac{x}{9}\left[\frac{1}{1-\{-(x/3)^2\}}\right] = \frac{x}{9}\sum_{n=0}^{\infty}\left[-\left(\frac{x}{3}\right)^2\right]^n = \frac{x}{9}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{9^n} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{9^{n+1}}$

The geometric series $\sum_{n=0}^{\infty}\left[-\left(\frac{x}{3}\right)^2\right]^n$ converges when $\left|-\left(\frac{x}{3}\right)^2\right| < 1 \quad \Leftrightarrow \quad \frac{|x^2|}{9} < 1 \quad \Leftrightarrow \quad |x|^2 < 9 \quad \Leftrightarrow \quad |x| < 3$, so $R = 3$ and $I = (-3, 3)$.

8. $f(x) = \frac{x}{2x^2+1} = x\left(\frac{1}{1-(-2x^2)}\right) = x\sum_{n=0}^{\infty}(-2x^2)^n$ or, equivalently, $\sum_{n=0}^{\infty}(-1)^n 2^n x^{2n+1}$. The series converges when $|-2x^2| < 1 \quad \Rightarrow \quad |x^2| < \frac{1}{2} \quad \Rightarrow \quad |x| < \frac{1}{\sqrt{2}}$, so $R = \frac{1}{\sqrt{2}}$ and $I = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

11. $f(x) = \frac{3}{x^2-x-2} = \frac{3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \quad \Rightarrow \quad 3 = A(x+1) + B(x-2)$. Let $x = 2$ to get $A = 1$ and $x = -1$ to get $B = -1$. Thus

$$\frac{3}{x^2-x-2} = \frac{1}{x-2} - \frac{1}{x+1} = \frac{1}{-2}\left(\frac{1}{1-(x/2)}\right) - \frac{1}{1-(-x)} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n - \sum_{n=0}^{\infty}(-x)^n$$
$$= \sum_{n=0}^{\infty}\left[-\frac{1}{2}\left(\frac{1}{2}\right)^n - 1(-1)^n\right]x^n = \sum_{n=0}^{\infty}\left[(-1)^{n+1} - \frac{1}{2^{n+1}}\right]x^n$$

We represented f as the sum of two geometric series; the first converges for $x \in (-2, 2)$ and the second converges for $(-1, 1)$. Thus, the sum converges for $x \in (-1, 1) = I$.

13. (a) $f(x) = \dfrac{1}{(1+x)^2} = \dfrac{d}{dx}\left(\dfrac{-1}{1+x}\right) = -\dfrac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$ [from Exercise 3]

$$= \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1} \quad \text{[from Theorem 2(i)]} \; = \sum_{n=0}^{\infty}(-1)^n (n+1) x^n \text{ with } R = 1.$$

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$].

(b) $f(x) = \dfrac{1}{(1+x)^3} = -\dfrac{1}{2}\dfrac{d}{dx}\left[\dfrac{1}{(1+x)^2}\right] = -\dfrac{1}{2}\dfrac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n (n+1) x^n\right]$ [from part (a)]

$$= -\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n (n+1) n x^{n-1} = \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n (n+2)(n+1) x^n \text{ with } R = 1.$$

(c) $f(x) = \dfrac{x^2}{(1+x)^3} = x^2 \cdot \dfrac{1}{(1+x)^3} = x^2 \cdot \dfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n (n+2)(n+1) x^n$ [from part (b)]

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n (n+2)(n+1) x^{n+2}$$

To write the power series with x^n rather than x^{n+2}, we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us $\dfrac{1}{2}\sum_{n=2}^{\infty}(-1)^n (n)(n-1) x^n$ with $R = 1$.