1. (a) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 8 > 1$, part (b) of the Ratio Test tells us that the series $\sum a_n$ is divergent.

(b) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0.8 < 1$, part (a) of the Ratio Test tells us that the series $\sum a_n$ is absolutely convergent (and therefore convergent).

(c) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test fails and the series $\sum a_n$ might converge or it might diverge.

4. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n^4}$ diverges by the Test for Divergence. $\lim_{n\to\infty}\frac{2^n}{n^4} = \infty$, so $\lim_{n\to\infty}(-1)^{n-1}\frac{2^n}{n^4}$ does not exist.

8. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(n+1)!}{100^{n+1}}\cdot\frac{100^n}{n!}\right] = \lim_{n\to\infty}\frac{n+1}{100} = \infty$, so the series $\sum_{n=1}^{\infty}\frac{n!}{100^n}$ diverges by the Ratio Test.

12. $\left|\frac{\sin 4n}{4^n}\right| \le \frac{1}{4^n}$, so $\sum_{n=1}^{\infty}\left|\frac{\sin 4n}{4^n}\right|$ converges by comparison with the convergent geometric series $\sum_{n=1}^{\infty}\frac{1}{4^n}$ $\left[|r| = \frac{1}{4} < 1\right]$.

Thus, $\sum_{n=1}^{\infty}\frac{\sin 4n}{4^n}$ is absolutely convergent.

21. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{n^2+1}{2n^2+1} = \lim_{n\to\infty}\frac{1+1/n^2}{2+1/n^2} = \frac{1}{2} < 1$, so the series $\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ is absolutely convergent by the Root Test.

30. By the recursive definition, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2+\cos n}{\sqrt{n}}\right| = 0 < 1$, so the series converges absolutely by the Ratio Test.

1. $\frac{1}{n+3^n} < \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$ for all $n \ge 1$. $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ is a convergent geometric series $\left[|r| = \frac{1}{3} < 1\right]$, so $\sum_{n=1}^{\infty}\frac{1}{n+3^n}$ converges by the Comparison Test.

7. Let $f(x) = \frac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on $[2,\infty)$, so we can apply the Integral Test.

Since $\int\frac{1}{x\sqrt{\ln x}}\,dx \begin{bmatrix} u = \ln x, \\ du = dx/x \end{bmatrix} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$, we find

$\int_2^{\infty}\frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty}\int_2^t\frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty}\left[2\sqrt{\ln x}\right]_2^t = \lim_{t\to\infty}\left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right) = \infty$. Since the integral diverges, the given series $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$ diverges.

8. $\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!} = \sum_{k=1}^{\infty} \frac{2^k}{(k+1)(k+2)}$. Using the Ratio Test, we get

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{2^{k+1}}{(k+2)(k+3)}\cdot\frac{(k+1)(k+2)}{2^k}\right| = \lim_{k\to\infty}\left(2\cdot\frac{k+1}{k+3}\right) = 2 > 1,$$ so the series diverges.

Or: Use the Test for Divergence.

15. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)[3(n+1)+2]}\cdot\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)}{n!}\right| = \lim_{n\to\infty}\frac{n+1}{3n+5} = \frac{1}{3} < 1,$$

so the series $\sum_{n=0}^{\infty}\frac{n!}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)}$ converges by the Ratio Test.

16. Using the Limit Comparison Test with $a_n = \frac{n^2+1}{n^3+1}$ and $b_n = \frac{1}{n}$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{n^2+1}{n^3+1}\cdot\frac{n}{1}\right) = \lim_{n\to\infty}\frac{n^3+n}{n^3+1} = \lim_{n\to\infty}\frac{1+1/n^2}{1+1/n^3} = 1 > 0.$$ Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ is also divergent.