30. If $p \le 0$, $\lim_{n\to\infty} \frac{\ln n}{n^p} = \infty$ and the series diverges, so assume $p > 0$. $f(x) = \frac{\ln x}{x^p}$ is positive and continuous and $f'(x) < 0$ for $x > e^{1/p}$, so f is eventually decreasing and we can use the Integral Test. Integration by parts gives

$$\int_1^\infty \frac{\ln x}{x^p}dx = \lim_{t\to\infty}\left[\frac{x^{1-p}\left[(1-p)\ln x - 1\right]}{(1-p)^2}\right]_1^t \quad (\text{for } p \ne 1) = \frac{1}{(1-p)^2}\left[\lim_{t\to\infty} t^{1-p}\left[(1-p)\ln t - 1\right] + 1\right],$$ which exists

whenever $1 - p < 0 \quad \Leftrightarrow \quad p > 1$. Thus, $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ converges $\quad \Leftrightarrow \quad p > 1$.

38. (a) $f(x) = \left(\frac{\ln x}{x}\right)^2$ is continuous and positive for $x > 1$, and since $f'(x) = \frac{2\ln x\,(1 - \ln x)}{x^3} < 0$ for $x > e$, we can apply the Integral Test. Using a CAS, we get $\int_1^\infty \left(\frac{\ln x}{x}\right)^2 dx = 2$, so the series also converges.

(b) Since the Integral Test applies, the error in $s \approx s_n$ is $R_n \le \int_n^\infty \left(\frac{\ln x}{x}\right)^2 dx = \frac{(\ln n)^2 + 2\ln n + 2}{n}$.

(c) By graphing the functions $y_1 = \frac{(\ln x)^2 + 2\ln x + 2}{x}$ and $y_2 = 0.05$, we see that $y_1 < y_2$ for $n \ge 1373$.

(d) Using the CAS to sum the first 1373 terms, we get $s_{1373} \approx 1.94$.

4. $\frac{n^3}{n^4 - 1} > \frac{n^3}{n^4} = \frac{1}{n}$ for all $n \ge 2$, so $\sum_{n=2}^{\infty} \frac{n^3}{n^4 - 1}$ diverges by comparison with $\sum_{n=2}^{\infty} \frac{1}{n}$, which diverges because it is a p-series with $p = 1 \le 1$ (the harmonic series).

12. $\frac{1 + \sin n}{10^n} \le \frac{2}{10^n}$ and $\sum_{n=0}^{\infty} \frac{2}{10^n} = 2\sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n$, so the given series converges by comparison with a constant multiple of a convergent geometric series.

14. $\frac{\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$, so $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ diverges by comparison with the divergent (partial) p-series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ $\quad \left[p = \frac{1}{2} \le 1\right]$.

20. $4^n > n$ for all $n \ge 1$ since the function $f(x) = 4^x - x$ satisfies $f(1) = 3$ and $f'(x) = 4^x \ln 4 - 1 > 0$ for $x \ge 1$, so $\frac{n + 4^n}{n + 6^n} < \frac{4^n + 4^n}{n + 6^n} < \frac{2 \cdot 4^n}{6^n} = 2\left(\frac{4}{6}\right)^n$, so the series $\sum_{n=1}^{\infty} \frac{n + 4^n}{n + 6^n}$ converges by comparison with $2\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$, which is a constant multiple of a convergent geometric series $\left[|r| = \frac{2}{3} < 1\right]$.

Or: Use the Limit Comparison Test with $a_n = \frac{n + 4^n}{n + 6^n}$ and $b_n = \left(\frac{2}{3}\right)^n$.

33. $\sum_{n=1}^{10} \frac{1}{\sqrt{n^4 + 1}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{17}} + \frac{1}{\sqrt{82}} + \cdots + \frac{1}{\sqrt{10{,}001}} \approx 1.24856$. Now $\frac{1}{\sqrt{n^4 + 1}} < \frac{1}{\sqrt{n^4}} = \frac{1}{n^2}$, so the error is

$$R_{10} \le T_{10} \le \int_{10}^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_{10}^t = \lim_{t\to\infty}\left(-\frac{1}{t} + \frac{1}{10}\right) = \frac{1}{10} = 0.1.$$

37. Since $\dfrac{d_n}{10^n} \le \dfrac{9}{10^n}$ for each n, and since $\sum\limits_{n=1}^{\infty} \dfrac{9}{10^n}$ is a convergent geometric series $\left(|r| = \frac{1}{10} < 1\right)$, $0.d_1 d_2 d_3 \ldots = \sum\limits_{n=1}^{\infty} \dfrac{d_n}{10^n}$ will always converge by the Comparison Test.

38. Clearly, if $p < 0$ then the series diverges, since $\lim\limits_{n\to\infty} \dfrac{1}{n^p \ln n} = \infty$. If $0 \le p \le 1$, then $n^p \ln n \le n \ln n \quad \Rightarrow$ $\dfrac{1}{n^p \ln n} \ge \dfrac{1}{n \ln n}$ and $\sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln n}$ diverges (Exercise 11.3.21), so $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^p \ln n}$ diverges. If $p > 1$, use the Limit Comparison Test with $a_n = \dfrac{1}{n^p \ln n}$ and $b_n = \dfrac{1}{n^p}$. $\sum\limits_{n=2}^{\infty} b_n$ converges, and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{1}{\ln n} = 0$, so $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^p \ln n}$ also converges. (Or use the Comparison Test, since $n^p \ln n > n^p$ for $n > e$.) In summary, the series converges if and only if $p > 1$.

43. $\lim\limits_{n\to\infty} n a_n = \lim\limits_{n\to\infty} \dfrac{a_n}{1/n}$, so we apply the Limit Comparison Test with $b_n = \dfrac{1}{n}$. Since $\lim\limits_{n\to\infty} n a_n > 0$ we know that either both series converge or both series diverge, and we also know that $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges $[p\text{-series with } p = 1]$. Therefore, $\sum a_n$ must be divergent.

7. $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (-1)^n \dfrac{3n-1}{2n+1} = \sum\limits_{n=1}^{\infty} (-1)^n b_n$. Now $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{3 - 1/n}{2 + 1/n} = \dfrac{3}{2} \ne 0$. Since $\lim\limits_{n\to\infty} a_n \ne 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

8. $b_n = \dfrac{n}{\sqrt{n^3+2}} > 0$ for $n \ge 1$. $\{b_n\}$ is decreasing for $n \ge 2$ since

$$\left(\frac{x}{\sqrt{x^3+2}}\right)' = \frac{(x^3+2)^{1/2}(1) - x \cdot \frac{1}{2}(x^3+2)^{-1/2}(3x^2)}{\left(\sqrt{x^3+2}\right)^2} = \frac{\frac{1}{2}(x^3+2)^{-1/2}[2(x^3+2) - 3x^3]}{(x^3+2)^1} = \frac{4 - x^3}{2(x^3+2)^{3/2}} < 0 \text{ for}$$

$x > \sqrt[3]{4} \approx 1.6$. Also, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{n/n}{\sqrt{n^3+2}/\sqrt{n^2}} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n + 2/n^2}} = 0$. Thus, the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{n}{\sqrt{n^3+2}}$ converges by the Alternating Series Test.

12. $b_n = \dfrac{e^{1/n}}{n} > 0$ for $n \ge 1$. $\{b_n\}$ is decreasing since $\left(\dfrac{e^{1/x}}{x}\right)' = \dfrac{x \cdot e^{1/x}(-1/x^2) - e^{1/x} \cdot 1}{x^2} = \dfrac{-e^{1/x}(1+x)}{x^3} < 0$ for $x > 0$. Also, $\lim\limits_{n\to\infty} b_n = 0$ since $\lim\limits_{n\to\infty} e^{1/n} = 1$. Thus, the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{e^{1/n}}{n}$ converges by the Alternating Series Test.

16. $\sin\left(\dfrac{n\pi}{2}\right) = 0$ if n is even and $(-1)^k$ if $n = 2k+1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{\sin(n\pi/2)}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!}$.

$b_n = \dfrac{1}{(2n+1)!} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} \dfrac{1}{(2n+1)!} = 0$, so the series converges by the Alternating Series Test.

24. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\,5^n}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)5^{n+1}} < \frac{1}{n\,5^n}$ and (ii) $\lim_{n\to\infty} \frac{1}{n\,5^n} = 0$, so the series is convergent. Now $b_4 = \frac{1}{4 \cdot 5^4} = 0.0004 > 0.0001$ and $b_5 = \frac{1}{5 \cdot 5^5} = 0.000064 < 0.0001$, so by the Alternating Series Estimation Theorem, $n = 4$. (That is, since the 5th term is less than the desired error, we need to add the first 4 terms to get the sum to the desired accuracy.)