2. (a) Using Equation 6, a power series expansion of f at 1 must have the form $f(1) + f'(1)(x-1) + \cdots$. Comparing to the given series, $1.6 - 0.8(x-1) + \cdots$, we must have $f'(1) = -0.8$. But from the graph, $f'(1)$ is positive. Hence, the given series is *not* the Taylor series of f centered at 1.

(b) A power series expansion of f at 2 must have the form $f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \cdots$. Comparing to the given series, $2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$, we must have $\frac{1}{2}f''(2) = 1.5$; that is, $f''(2)$ is positive. But from the graph, f is concave downward near $x = 2$, so $f''(2)$ must be negative. Hence, the given series is *not* the Taylor series of f centered at 2.

3. Since $f^{(n)}(0) = (n+1)!$, Equation 7 gives the Maclaurin series

$\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!}x^n = \sum_{n=0}^{\infty}(n+1)x^n$. Applying the Ratio Test with $a_n = (n+1)x^n$ gives us

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = |x| \lim_{n\to\infty}\frac{n+2}{n+1} = |x| \cdot 1 = |x|$. For convergence, we must have $|x| < 1$, so the radius of convergence $R = 1$.

4. Since $f^{(n)}(4) = \dfrac{(-1)^n\, n!}{3^n(n+1)}$, Equation 6 gives the Taylor series

$\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n\, n!}{3^n(n+1)\, n!}(x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)}(x-4)^n$, which is the Taylor series for f centered at 4. Apply the Ratio Test to find the radius of convergence R.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-4)^{n+1}}{3^{n+1}(n+2)} \cdot \frac{3^n(n+1)}{(-1)^n(x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)(x-4)(n+1)}{3(n+2)}\right|$$
$$= \frac{1}{3}|x-4| \lim_{n\to\infty}\frac{n+1}{n+2} = \frac{1}{3}|x-4|$$

For convergence, $\frac{1}{3}|x-4| < 1 \quad \Leftrightarrow \quad |x-4| < 3$, so $R = 3$.

7.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin \pi x$	0
1	$\pi \cos \pi x$	π
2	$-\pi^2 \sin \pi x$	0
3	$-\pi^3 \cos \pi x$	$-\pi^3$
4	$\pi^4 \sin \pi x$	0
5	$\pi^5 \cos \pi x$	π^5
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\sin \pi x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots \\ &= 0 + \pi x + 0 - \frac{\pi^3}{3!}x^3 + 0 + \frac{\pi^5}{5!}x^5 + \cdots \\ &= \pi x - \frac{\pi^3}{3!}x^3 + \frac{\pi^5}{5!}x^5 - \frac{\pi^7}{7!}x^7 + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} x^{2n+1}\end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\pi^{2n+3}\,x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{\pi^{2n+1}\,x^{2n+1}}\right| = \lim_{n\to\infty}\frac{\pi^2\,x^2}{(2n+3)(2n+2)}$$

$= 0 < 1$ for all x, so $R = \infty$.

10.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^x	0
1	$(x+1)e^x$	1
2	$(x+2)e^x$	2
3	$(x+3)e^x$	3
$\vdots$	$\vdots$	$\vdots$

$$xe^x = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{n}{n!}x^n = \sum_{n=1}^{\infty}\frac{n}{n!}x^n = \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!}.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x|^{n+1}}{n!}\cdot\frac{(n-1)!}{|x|^n}\right] = \lim_{n\to\infty}\frac{|x|}{n} = 0 < 1 \text{ for all } x,$$

so $R = \infty$.

21. If $f(x) = \sin \pi x$, then $f^{(n+1)}(x) = \pm\pi^{n+1}\sin \pi x$ or $\pm\pi^{n+1}\cos \pi x$. In each case, $\left|f^{(n+1)}(x)\right| \le \pi^{n+1}$, so by Formula 9 with $a = 0$ and $M = \pi^{n+1}$, $|R_n(x)| \le \dfrac{\pi^{n+1}}{(n+1)!}\,|x|^{n+1} = \dfrac{|\pi x|^{n+1}}{(n+1)!}$. Thus, $|R_n(x)| \to 0$ as $n \to \infty$ by Equation 10. So $\lim_{n\to\infty} R_n(x) = 0$ and, by Theorem 8, the series in Exercise 7 represents $\sin \pi x$ for all x.

33. $\cos x = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{(2n)!} \Rightarrow \cos\left(\frac{1}{2}x^2\right) = \sum_{n=0}^{\infty}(-1)^n\dfrac{\left(\frac{1}{2}x^2\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{4n}}{2^{2n}\,(2n)!}$, so

$f(x) = x\cos\left(\frac{1}{2}x^2\right) = \sum_{n=0}^{\infty}(-1)^n\dfrac{1}{2^{2n}(2n)!}x^{4n+1}$, $R = \infty$.

34. $\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1} \Rightarrow \tan^{-1}(x^3) = \sum_{n=0}^{\infty}(-1)^n\dfrac{(x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{6n+3}}{2n+1}$, so

$x^2\tan^{-1}(x^3) = \sum_{n=0}^{\infty}(-1)^n\dfrac{1}{2n+1}x^{6n+5}$; $\left|x^3\right| < 1 \Leftrightarrow |x| < 1$, so $R = 1$.

58. $\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots\right) - x}{x^3} = \lim_{x \to 0} \frac{\frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots}{x^3} = \lim_{x \to 0} \left(\frac{1}{3} + \frac{2}{15}x^2 + \cdots\right) = \frac{1}{3}$

since power series are continuous functions.

63. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{\left(-x^4\right)^n}{n!} = e^{-x^4}$, by (11).

66. $\sum_{n=0}^{\infty} \frac{3^n}{5^n\, n!} = \sum_{n=0}^{\infty} \frac{(3/5)^n}{n!} = e^{3/5}$, by (11).