Practice Math 25a Midterm #1 Solutions

Davis Lazowski

Problem 1. Let V be an n–dimensional vector space and let $f\colon V \to V$ be a linear map such that $\ker f = \operatorname{im} f$. Show that n is even.

Solution. In this case $\dim \ker f = \dim \operatorname{im} \mathrm{f}$. By rank nullity,

$$n = \dim V = \dim \ker f + \dim \ker \operatorname{im} = 2 \dim \ker f,$$

so that n is even, as required. (DL)

Problem 2. Let V be a finite dimensional vector space and let $f\colon V \to V$ be a linear function. Suppose that *any* choice of basis for V gives the same matrix representation. Prove that $f = \alpha \cdot \mathrm{id}$ for some scalar α.

Solution. Let $v_1, v_2, v_3 ... v_n$ a basis. Then

$$T(v_1) = \alpha_{11} v_1 + \alpha_{21} v_2 + \cdots + \alpha_{n1} v_n$$

Also, $-v_1, v_2, v_3 ... v_n$ is a basis. So

$$T(-v_1) = -\alpha_{11} v_1 + \alpha_{21} v_2 + \cdots + \alpha_{n1} v_n$$

By $-1T(v_1) = T(-v_1)$,

$$T(-v_1) = -\alpha_{11} v_1 - \alpha_{21} v_2 - \cdots - \alpha_{n1} v_n$$

Since these are equal we have that,

$$2T(-v_1) = -2\alpha_{11} v_1$$
$$T(v_1) = \alpha_{11} v_1$$

By linear dependence, therefore $\alpha_{21} ... \alpha_{n1}$ are zero. Therefore by induction, T is diagonal.

Therefore using the basis $(v_1, v_2 .... v_n)$:

$$T(v_1) = \alpha_{11} v_1$$
$$T(v_2) = \alpha_{22} v_2$$

By transposition, $(v_2, v_1 ... v_n)$ is a basis:

$$T(v_2) = \alpha_{11} v_2$$
$$T(v_1) = \alpha_{22} v_1$$

Therefore, $\alpha_{11} = \alpha_{22}$. By induction, $\alpha_{jj} = \alpha_{j'j'}$. Therefore this matrix is $T = \alpha \mathrm{id}$, as desired. (DL)

Problem 3. Suppose that W is a complex vector space and $f\colon\ W \to W$ has no eigenvalues. Prove that every subspace of W invariant under f is either 0 or ∞–dimensional.

Solution. Suppose $\dim W = n, 0 < n < \infty$. Let $w \in W$. Consider the list $w, fw, f^2w, f^3w, \dots f^n w$. This is a list of $n+1$ vectors, so is linearly dependent. So

$$0 = a_0 w + a_1 f w + \dots + a_n f^n w$$
$$0 = (a_0 + a_1 x + \dots a_n x^n)(f)(w)$$

Because this polynomial is over the complex numbers, we can factor this expression:

$$0 = (f - r_1)(f - r_2) \dots (f - r_n) w$$

Because the operator $p(f)$ is not injective (and in particular $p(f_{|W})$ is not injective) one of these must not be also. So there must be $v \in W : (f - r_j)v = 0$. Therefore, $fv = r_j v$, so there is an eigenvalue. Therefore, $\dim W$ must be 0 or ∞. (DL)

Problem 4. Let $f\colon\ K^2 \to K^2$ act by $f(x, y) = (y, x)$. Is f diagonalizable? If so, diagonalize it. If not, argue why not.

Solution. f has eigenvector $(1, 1)$. If f is diagonalisable, it has a second eigenvector in $\langle(1, 1)\rangle^\perp$. This is a one dimensional subspace, which is spanned by $(1, -1)$, and we see $f(1, -1) = k(-1, 1)$ is satisfied by $k = -1$. Therefore, f is diagonal in the basis

$$\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right),$$

with matrix expression

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ (DL)

Problem 5. Suppose V is finite-dimensional, $f\colon\ V \to V$ is a linear function, and $U \leq V$ is a subspace. Prove that U and $U^\perp$ are both invariant under f if and only if $P_U f = f P_U$.

Solution. Suppose $U, U^\perp$ are invariant under f. Let $u \in U, w \in U^\perp$. Then we can write $v \in V$ as $v = u + w$. Then because $f(u) \in U, f(w) \in U^\perp$,

$$f(P_U(u + w)) = f(P_U(u)) + f(P_U(w)) = f(u) + 0 = f(u)$$
$$P_U(f(u + w)) = P_U(f(u)) + P_U(f(w)) = f(u)$$

So $P_U f = f P_U$, as required.

Second, suppose $P_U f = f P_U$. Let $u \in U$. We can write $f(u) = \tilde{u} + \tilde{w}$, with $\tilde{u} \in U, \tilde{w} \in U^\perp$.

Then

$$\begin{aligned} f P_U(u) &= f(u) = \tilde{u} + \tilde{w} \\ P_U f(u) &= P_U(\tilde{u} + \tilde{w}) = \tilde{u} \\ \tilde{u} + \tilde{w} &= \tilde{u} \implies \tilde{w} = 0 \end{aligned}$$

So that f is invariant under U.

Now, let $w \in U^\perp$. Let $f(w) = u_o + w_o$, with $u_o \in U, w_o \in U^\perp$.

Then

$$\begin{aligned} f P_U(w) &= f(0) = 0 \\ P_U f(w) &= P_U(u_o + w_o) = u_0 \\ &\implies 0 = u_0 \end{aligned}$$

So that f is invariant under $U^\perp$ also. (DL)

Practice Math 25a Midterm #2 Solutions

Eric Peterson

Problem 1. Let (a, b) and (c, d) be two vectors in $\mathbb{R}^2$.

1. Show that (a, b) and (c, d) are linearly dependent if and only if $ad - bc = 0$.
2. Consider the map $\varphi\colon \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2) \to \mathbb{R}$ described by
$$\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc.$$
Prove or disprove that φ is a linear function.

Solution. 1. If there is a linear dependence, then $s(a, b) = t(c, d)$ for some $s, t \in \mathbb{R}$ not both zero. By noticing that
$$\varphi((a, b), (c, d)) = ad - bc = -(cb - da) = -\varphi((c, d), (a, b)),$$
we see that the condition that φ is zero or nonzero is invariant under swapping the vectors. So, we may as well assume that t is nonzero and express (c, d) as
$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{s}{t} \cdot \begin{pmatrix} a \\ b \end{pmatrix}.$$
We thus calculate
$$\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = a(sb/t) - b(sa/t) = 0.$$
Conversely, suppose that $ad - bc = 0$. Since (a, b) is not the zero vector, one of a or b is nonzero.

- Suppose $a \neq 0$, so that we can solve to get $d = bc/a$. This expresses $(c, d) = (c, bc/a) = c/a \cdot (a, b)$.
- Otherwise, suppose $b \neq 0$, so that we can solve to get $c = ad/b$. This expresses $(c, d) = (ad/b, d) = d/b \cdot (a, b)$.

2. φ fails to be a linear function. Consider the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

φ applied to either matrix gives 0. However, φ applied to their sum gives 1.

Alternatively, $\varphi(k \cdot M) = k^2 \cdot \varphi(M)$ for a scalar k violates the usual scalar multiplication condition. (ECP)

Problem 2. Let V be a vector space which decomposes as a direct sum of two subspaces $U, U' \leq V$, and set $N = \{\varphi \in V^* \mid \varphi(U) = 0\}$. Show that N is a subspace of V^* and that V^*/N is isomorphic to U^*.

Solution. To see that N is a subspace, we note that it is closed under sums — for $\varphi, \psi \in N$, we have $(\varphi + \psi)(U) \subseteq \varphi(U) + \psi(U) = 0 + 0 = 0$ — and under scalar multiplication — for $\varphi \in N$, $(k \cdot \varphi)(U) = k \cdot \varphi(U) = k \cdot 0 = 0$. Now, consider the inclusion $i \colon U \to V$ and its dual $i^* \colon V^* \to U^*$. First, i^* is surjective: any functional on U can be lifted to a functional on V by extending by zero on the complement U'. Hence, there is a factorization

$$V^* \to V^*/\ker i^* \xrightarrow{\cong} U^*,$$

and we are left with showing $\ker i^* = N$. Expand the defintion:

$$\ker i^* = \{\varphi \in V^* \mid i^* = 0\} = \{\varphi \in V^* \mid \varphi \circ i = 0\} = \{\varphi \in V^* \mid \varphi(U) = 0\} = N. \qquad \text{(ECP)}$$

Problem 3. Let V be a complex vector space of finite dimension and let $f \colon V \to V$ be a linear function. Prove there exists a basis $(v_1, \ldots, v_n)$ of V such that the matrix presenting f is upper-triangular. (Feel free to assume that f admits an eigenvector.)

Solution. Begin by selecting an eigenvector v of f, which spans an invariant 1–dimensional subspace $U = \langle v \rangle$. By induction, the operator f/U admits a upper-triangularization by a basis $(w_2, \ldots, w_n)$ of V/U. Lift these to vectors $w_j = v_j + U$ in V. The new list $(v_1, \ldots, v_n)$ forms a basis for V, as they span V and the list has the correct length. The behavior of f/U on w_j (namely: $f/U(w_j) \in \operatorname{span}\{w_2, \ldots, w_n\}$) shows that the behavior of f on v_j has the upper triangularity property: $f(v_j) \in \operatorname{span}\{v_1, \ldots, v_j\}$. (ECP)

Problem 4. Prove that every operator on a finite-dimensional nonzero real vector space has an invariant subspace of dimension 1 or 2.

Solution. As in the proof that complex operators admit eigenvectors (i.e., 1–dimensional invariant subspaces), consider a vector $v \neq 0$ as well as the list

$$(v, fv, f^2 v, \ldots, f^n v)$$

for $n = \dim V$. There is necessarily a linear dependence among this list, which we consider as follows:

$$\begin{aligned} a_0 v + a_1 f v + a_2 f^2 v + \cdots + a_n f^n v &= 0 \\ (a_0 + a_1 f + \cdots + a_n f^n)(v) &= 0 \\ c(f - r_1) \cdots (f - r_m)((f - h_1)^2 + k_1^2) \cdots ((f - h_\ell)^2 + k_\ell^2)(v) &= 0, \end{aligned}$$

where we have used the factorization theorem for real polynomials at the last line. Since this whole product annihilates v, one of the factors must fail to be injective. In the case that $(f - r_j)(w) = 0$, we have a linear dependence in the list (w, fw), which gives an invariant subspace of dimension 1, spanned by w. Alternatively, in the case $((f - h_j)^2 + k_j^2)(w) = 0$, we have a linear dependence in the list (w, fw, ffw), which gives an invariant subspace of dimension 2, spanned by w and fw. (ECP)

Problem 5. Suppose $u, v \in V$ and $\|u\| = \|v\| = 1$ and $\langle u, v \rangle = 1$. Prove that $u = v$.

Solution. We want to show $u - v = 0$, so we calcuate $\|u - v\|$.

$$\begin{aligned} \|u - v\|^2 &= \langle u - v, u - v \rangle \\ &= \langle u, u \rangle + \langle v, v \rangle - \langle u, v \rangle - \langle v, u \rangle \\ &= \|u\| + \|v\| - 1 - 1 = 0. \end{aligned}$$

Since $\|u - v\|^2 = 0$ only for the zero vector, we are done. (ECP)

Math 25a Midterm Solutions

Eric Peterson

Problem 1. Suppose $v_1, \ldots, v_m$ is a linearly independent set of vectors in V, and suppose that $w \in V$ is another vector. Show that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}\{v_1, \ldots, v_m\}$.

Solution. Suppose there is a nonzero linear dependence:

$$k_1(v_1 + w) + \cdots + k_m(v_m + w) = 0.$$

Rearrange this for w:

$$k_1 w + \cdots + k_m w = -k_1 v_1 + \cdots + -k_m v_m.$$

If $k_1 + \cdots + k_m = 0$, this gives a linear dependence among the v_j, which we know to be *independent*. Hence $k_1 + \cdots + k_m \neq 0$, and we can divide by it:

$$w = \frac{k_1}{k_1 + \cdots + k_m} v_1 + \cdots + \frac{k_m}{k_1 + \cdots + k_m} v_m \in \operatorname{span}\{v_1, \ldots, v_m\}. \qquad \text{(ECP)}$$

Problem 2. For a subspace $U \leq V$, recall that a functional $\varphi \in V^*$ is said to *annihilate* U if $\varphi(U) = 0$. The set of functionals satisfying this condition form a subspace U^0 of V^*. Supposing that V is finite dimensional, prove

$$\dim V = \dim U + \dim U^0.$$

Solution. Consider the inclusion map $i\colon U \to V$ and its dual $i^*\colon V^* \to U^*$. The Fundamental Theorem of Linear Algebra gives

$$\dim V^* = \dim \ker i^* + \dim \operatorname{im} i^*.$$

First, note that i^* is surjective: by picking a complement U' to U, we can lift any functional on U to a function on V by extending by 0 on U'. Hence, $\operatorname{im} i^* = U^*$ and $\dim \operatorname{im} i^* = \dim U^*$. Second, note that $\ker i^* = U^0$:

$$\ker i^* = \{\varphi \in V^* \mid i^*\varphi = 0\} = \{\varphi \in V^* \mid \varphi \circ i = 0\} = \{\varphi \in V^* \mid \varphi(U) = 0\} = U^0.$$

Hence, $\dim \ker i^* = \dim U^0$. Tying these together gives

$$\dim V^* = \dim U^* + \dim U^0.$$

Finally, the dimensions of finite dimensional spaces and their duals agree, so

$$\dim V = \dim U + \dim U^0. \qquad \text{(ECP)}$$

Solution. Alternatively, we can choose a basis $v_1, \dots, v_j$ of U and extend it to a basis $v_1, \dots, v_j, v_{j+1}, \dots, v_n$ of V. This gives rise to a dual basis $v_1^*, \dots, v_n^*$ of V^*, and we would like to show that $v_{j+1}^*, \dots, v_n^*$ gives a basis for the annihilator subspace U^0. We already know that this list is linearly independent, so we are left with showing

$$\operatorname{span}\{v_{j+1}^*, \dots, v_n^*\} = U^0.$$

($\subseteq$:) The functionals v_i^* are determined by

$$v_i^*(v_k) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise,} \end{cases}$$

which in turn gives the formula

$$v_i^*(v) = v_i^*(a_1 v_1 + \cdots + a_n v_n) = a_i.$$

It follows that

$$v_{>j}^*(u) = v_{>j}^*(a_1 v_1 + \cdots + a_j v_j) = 0,$$

so that $v_{>j}^* \in U^0$.

($\supseteq$:) Because the functionals above give a basis for V^*, an arbitrary functional $\varphi \in V^*$ can be expressed as

$$\varphi = c_1 v_1^* + \cdots + c_n v_n^*.$$

We can compute some of these coefficients: if $\varphi \in U^0$, then $\varphi(u_{\leq j}) = 0$ and $\varphi(u_{\leq j}) = c_{\leq j}$, hence $c_{\leq j} = 0$. This shows $\varphi \in \operatorname{span}\{v_{j+1}^*, \dots, v_n^*\}$.

(ECP)

Problem 3. Let M be an $n \times n$ matrix with real entries, and let v be an eigenvector of M with eigenvalue λ.

1. Prove that for all $k \geq 1$, λ^k is an eigenvalue of M^k. Describe an associated eigenvector.
2. Suppose that M is furthermore nilpotent, meaning that $M^r = 0$ for some $r \gg 0$. Prove that 0 is the only eigenvalue of M.

Solution.

1. The base case of the induction is given in the problem hypothesis. Then, consider

$$M^k(v) = M(M^{k-1}(v)) = M(\lambda^{k-1}v) = \lambda^{k-1}Mv = \lambda^{k-1}\lambda v = \lambda^k v.$$

Hence, v is an eigenvector of M^k with eigenvalue λ^k.

2. If λ is an eigenvalue of M, then λ^r is an eigenvalue of M^r. However, since $M^r = 0$, its only eigenvalues are 0, hence $\lambda^r = 0$. This is only soluable if λ itself is zero (since the product, hence power, of nonzero numbers is nonzero). (ECP)

Problem 4. Let $f, g\colon V \to V$ be two linear functions. Show that $f \circ g$ and $g \circ f$ must have the same eigenvalues. **This was misprinted on the exam. V should additionally be assumed to be finite-dimensional.**

Solution. For $\lambda \neq 0$ is an eigenvalue of $f \circ g$, choose an associated eigenvector v with $fgv = \lambda v$. Then, set $w = gv$, which is nonzero because $fgv = \lambda v$ is nonzero, and consider gf applied to w:

$$gfw = gfgv = g(\lambda v) = \lambda gv = \lambda w.$$

Similarly, if $\lambda \neq 0$ were instead an eigenvalue of $g \circ f$ with associated eigenvector v satisfying $gfv = \lambda v$, we would set $w = fv \neq 0$ and consider fg applied to w:

$$fgw = fgfv = f(\lambda v) = \lambda fv = \lambda w.$$

In order to deal with the case $\lambda = 0$, we need the misprinted additional assumption. If $fgv = 0 \cdot v = 0$ and $gv = w \neq 0$, we can proceed as above to exhibit w as an eigenvector of gf with eigenvalue 0. However, if $w = 0$, then we need to form some other candidate vector w' for which $gfw' = 0$. If f is not injective, then f must have a nontrivial kernel, and we can pick a nontrivial element of its kernel to use as w'. If f is injective, then *by finite-dimensionality* it is also surjective, and hence v has a preimage w' satisfying $f(w') = v$. This element then satisfies $gfw' = gv = 0$. The other inclusion is shown identically, reversing the appearances of the fs and gs in this argument.[1] (ECP)

Problem 5. Suppose $f\colon V \to V$ is a linear function on a finite-dimensional inner product space such that $\|f(v)\| \leq \|v\|$ for every $v \in V$. Prove that $f - \sqrt{2} \cdot \mathrm{id}$ is invertible.

Solution. The contrapositive is easier to prove. If $f - \sqrt{2}$ fails to be invertible, then there is a nonzero element v in its kernel, which is an eigenvector of f of eigenvalue $\sqrt{2}$. This element satisfies $\|fv\| = \|\sqrt{2}v\| = \sqrt{2}\|v\|$, which shows $\|fv\| \not\leq \|v\|$ for this choice of v. (ECP)

Solution. Alternatively, for a nonzero vector v we have $\|\sqrt{2}v\| = \sqrt{2} \cdot \|v\|$ and $\|fv\| \leq \|v\|$. The triangle inequality forces

$$\|fv - \sqrt{2}v - fv\| \leq \|fv - \sqrt{2}v\| + \|fv\|,$$

or

$$\|\sqrt{2}v\| - \|fv\| \leq \|\sqrt{2}v - fv\|.$$

Then, the assumption gives

$$(\sqrt{2} - 1)\|v\| = \|\sqrt{2}v\| - \|v\| \leq \|\sqrt{2}v - fv\|,$$

so that $\|v\| \neq 0$ forces $\|(\sqrt{2} - f)v\| \neq 0$. This operator is therefore injective, hence invertible. (ECP)

[1] Without this assumption, consider integration and differentiation of polynomials. Differentiation has a kernel, the constant polynomials, but integration is injective but not surjective — it's even a right-inverse to differentiation. This means that $\int dx \circ d/dx$ has a kernel, hence an eigenvector of weight 0, but $d/dx \circ \int dx = \mathrm{id}$ has no kernel, hence no eigenvectors of weight 0.

Problem 6. Suppose $u, v \in V$ for an inner product space V. Prove that $\|au + bv\| = \|bu + av\|$ for all $a, b \in \mathbb{R}$ if and only if $\|u\| = \|v\|$.

Solution. Expand the two inner product formulas:

$$\begin{aligned}
\|au + bv\| &= \langle au + bv, au + bv \rangle = \langle au, au \rangle + \langle au, bv \rangle + \langle bv, au \rangle + \langle bv, bv \rangle \\
&= a^2\|u\|^2 + b^2\|v\|^2 + ab\,\mathrm{Re}\langle u, v \rangle, \\
\|bu + av\| &= \langle bu + av, bu + av \rangle = \langle bu, bu \rangle + \langle bu, av \rangle + \langle av, bu \rangle + \langle av, av \rangle \\
&= a^2\|v\|^2 + b^2\|u\|^2 + ab\,\mathrm{Re}\langle u, v \rangle.
\end{aligned}$$

($\Rightarrow$) Specialize the above expressions to $a = 1$ and $b = 0$ to get $\|u\|^2 = \|v\|^2$.

($\Leftarrow$) If $\|u\| = \|v\|$, then the first two terms of each line match. The third term always matches, independent of assumption on u and v. (ECP)

Scratch work.

Scratch work.