Homework #9

Math 25a

Due: November 9nd, 2016

Guidelines:

- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one (though you should), a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!)

1 For submission to Thayer Anderson

Problem 1.1. Suppose for $f\colon V \to V$ that $v \in V$ and $m \in \mathbb{N}$ satisfy $f^m(v) = 0$ but $f^{m-1}(v) \neq 0$. Prove that the following list is linearly independent:

$$v, fv, f^2 v, \dots, f^{m-1} v.$$

Problem 1.2. Suppose $f\colon V \to W$ is a linear function between inner product spaces. Show that $f^* f$ is a positive operator on V and ff^* is a positive operator on W.

Problem 1.3. Suppose $f\colon V \to V$ is a linear function on a finite-dimensional inner product space. Define a new pairing by

$$\langle u, v \rangle_f = \langle fu, v \rangle.$$

Show that $\langle -, - \rangle_f$ is an inner product on V if and only if f is an invertible positive operator.

2 For submission to Davis Lazowski

Problem 2.1. Suppose $f, g\colon V \to V$ are linear operators and suppose that fg is nilpotent. Prove that gf is also nilpotent.

Problem 2.2. Suppose that $f\colon V \to V$ is a linear function on an inner product space, and suppose that there exists an orthonormal basis $e_1, \dots, e_n$ of V such that $\|fe_j\| = 1$ for each j. Either show that f must be an isometry or give a counterexample.

Problem 2.3. Fix vectors $u, x \in V$ in a finite-dimensional vector space V with $u \neq 0$. Consider an operator $f\colon V \to V$ defined by

$$f(v) = \langle v, u \rangle \cdot x$$

(as in one of the summands in singular value decomposition). Prove the following:

$$\sqrt{f^* f}(v) = \frac{\|x\|}{\|u\|} \langle v, u \rangle \cdot u.$$

Problem 2.4. Suppose $f\colon V \to V$ has singular value decomposition given by

$$f(v) = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n$$

for $s_1, \ldots, s_n$ the singular values of f and $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ orthonormal bases of V. Prove the following effects:

1. $f^*(v) = s_1 \langle v, f_1 \rangle e_1 + \cdots + s_n \langle v, f_n \rangle e_n$.
2. $f^* f(v) = s_1^2 \langle v, e_1 \rangle e_1 + \cdots + s_n^2 \langle v, e_n \rangle e_n$.
3. $\sqrt{f^* f}(v) = s_1 \langle v, e_1 \rangle e_1 + \cdots + s_n \langle v, e_n \rangle e_n$.
4. Lastly, suppose f is invertible. Show also $f^{-1}(v) = s_1^{-1} \langle v, f_1 \rangle e_1 + \cdots + s_n^{-1} \langle v, f_n \rangle e_n$.

3 For submission to Handong Park

Problem 3.1. Prove or give a counterexample: the set of nilpotent operators on V is a vector subspace of $\mathcal{L}(V, V)$.

Problem 3.2. For $f\colon V \to V$ a linear operator on a finite-dimensional inner product space, write $s_{\min}$ for its smallest singular value and $s_{\max}$ for its largest singular value.

1. Prove the inequalities $s_{\min} \|v\| \leq \|fv\| \leq s_{\max} \|v\|$.
2. For any eigenvalue λ of f, show $s_{\min} \leq |\lambda| \leq s_{\max}$.
3. Let $g\colon V \to V$ be another linear operator with minimum and maximum singular values $t_{\min}$ and $t_{\max}$ respectively. Show that the maximum singular value of the composite gf is bounded above by $s_{\max} \cdot t_{\max}$ and that the maximum singular value of the sum $g + f$ is bounded above by $s_{\max} + t_{\max}$.

Problem 3.3. Suppose that V is a finite-dimensional inner product space, $f\colon V \to V$ is a linear operator, $g\colon V \to V$ is an isometry, and $h\colon V \to V$ a positive operator satisfying $f = g \circ h$. Show that $h = \sqrt{f^* f}$.

4 For submission to Rohil Prasad

Problem 4.1. Suppose $f\colon V \to V$ is a linear operator on a finite-dimensional inner product space. Show that $\dim \operatorname{im} f$ equals the number of nonzero singular values of f.

Problem 4.2. Last week in Problem 4.2, you considered the inner product space of continuous functions on $[-\pi, \pi]$, as well as the subspace

$$U_n = \operatorname{span}\{1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx\}$$

and the double-derivative operator $D^2\colon U_n \to U_n$. Show that $-D^2$ is a positive operator.

Problem 4.3. Define $f\colon \mathbb{C}^3 \to \mathbb{C}^3$ by $f(z_1, z_2, z_3) = (z_2, z_3, 0)$. Prove that f has no square-root.