Homework #8

Math 25a

Due: November 2nd, 2016

Guidelines:

- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one (though you should), a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!)

1 For submission to Thayer Anderson

Problem 1.1. Prove that a normal operator on a finite-dimensional complex inner product space is self-adjoint if and only if all its eigenvalues are real.

Problem 1.2. Give an example of a real inner product space V and an operator $f\colon V \to V$ such that $(f-h)^2 + k^2$ is *not* invertible for some $h, k \in \mathbb{R}$ and $k > 0$. (Note: this shows that the self-adjointness hypothesis is necessary to find an eigenvalue for f, which we used as input for the real spectral theorem.)

Problem 1.3. Prove or give a counterexample: if there exists an orthonormal basis $(e_1, \ldots, e_n)$ such that an operator $f\colon V \to V$ satisfies $\|f(e_j)\| = \|f^*(e_j)\|$, then f is normal.

2 For submission to Davis Lazowski

Problem 2.1. Take V to be a finite-dimensional inner product space and $f\colon V \to V$ a linear operator on it. Show that λ is an eigenvalue of f exactly when $\overline{\lambda}$ is an eigenvalue of the adjoint map f^*. (Note: because f is *not* assumed normal, the associated eigenvectors may differ.)

Problem 2.2. Suppose that $f\colon V \to W$ is a linear map between real inner product spaces. Recall the definition of the adjoint map from class:

$$\begin{tikzcd}
V^* \arrow[r, bend left, red] & V \arrow[l, bend left] \\
W^* \arrow[u, red, "\text{the dual map, } f^*"] \arrow[r, bend left] & W \arrow[l, bend left, red] \arrow[u, dotted, "\text{the adjoint map, } f^*"']
\end{tikzcd}$$

where, e.g., $W \to W^*$ is the isomorphism $w \mapsto \langle -, w \rangle$. Using this definition, check the claim

$$\langle f(v), w \rangle_W = \langle v, f^*(w) \rangle_V.$$

Problem 2.3. Suppose $f\colon V \to V$ is a self-adjoint operator on a finite-dimensional inner product space. Suppose that for $\lambda \in \mathbb{R}$ and $\varepsilon > 0$, there exists a $v \in V$ with $\|v\| = 1$ and

$$\|f(v) - \lambda \cdot v\| < \varepsilon.$$

Show that f has an eigenvalue λ' such that $|\lambda - \lambda'| < \varepsilon$.

3 For submission to Handong Park

Problem 3.1. Suppose that V is a finite-dimensional real vector space, and suppose that $f\colon V \to V$ is a linear operator on V. Prove that V has a basis consisting of eigenvectors of f if and only if there is an inner product on V for which f is self-adjoint.

Problem 3.2. Suppose that V is a finite-dimensional inner product space, that $P\colon V \to V$ is a linear function, and that $P \circ P = P$.

1. Show that if $\operatorname{im} P \perp \ker P$, then there exists a subspace $U \le V$ such that $P = P_U$.
2. Show that if $\|Pv\| \le \|v\|$ for all $v \in V$, then there exists a subspace $U \le V$ such that $P = P_U$.
3. Show that P is self-adjoint if and only if there is a subspace $U \le V$ with $P = P_U$.

Problem 3.3. Suppose that $f, g\colon V \to V$ are both self-adjoint. Show that $f \circ g$ is self-adjoint if and only if $f \circ g = g \circ f$.[1]

4 For submission to Rohil Prasad

Problem 4.1. Suppose that f is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of f. Show then that

$$f^2 - 5f + 6 = 0.$$

Show additionally that this conclusion fails if f is not self-adjoint.

Problem 4.2. Consider the vector space $V = C[-\pi, \pi]$ of continuous functions with signature $[-\pi, \pi] \to \mathbb{R}$, which has an inner product specified by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) dx.$$

For an $n \in \mathbb{N}_{>0}$, construct a subspace

$$U_n = \operatorname{span}\{1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx\}.$$

1. Define $D\colon U_n \to U_n$ by differentiation: $Df = f'$. Show that $D^* = -D$.
2. Is D normal? Is it self-adjoint? What about $D^{\circ 2}$?

Problem 4.3. Recall that our proof of the real spectral theorem rests primarily on knowing that the perpendicular subspace of an invariant subspace is itself invariant. Rewrite our proof of the real spectral theorem to apply to normal operator on a complex vector space.

[1] The functions f and g are said to "commute".