Homework #8 Solutions

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1 For submission to Thayer Anderson

Problem 1.1. Prove that a normal operator on a finite-dimensional complex inner product space is self-adjoint if and only if all its eigenvalues are real.

Solution. Suppose that f is self-adjoint. Suppose that v is an eigenvector of f with associated eigenvalue λ. Then

$$\langle fv, v\rangle = \langle \lambda v, v\rangle = \lambda\langle v, v\rangle = \lambda||v||^2.$$

Applying the self-adjoint condition, we obtain,

$$\langle fv, v\rangle = \langle v, fv\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\langle v, v\rangle = \bar{\lambda}||v||^2$$

The vector v is non-zero and therefore $||v||^2$ is non-zero. Therefore,

$$\begin{aligned} \lambda||v||^2 &= \bar{\lambda}||v||^2 \\ \Rightarrow \lambda &= \bar{\lambda}. \end{aligned}$$

This implies that λ is real.

For the next direction, suppose that f has all real eigenvalues. We are given that f is normal and therefore orthonormally diagonalizable. In its diagonal representation, all the entries of the matrix encoding of f will be real and on the diagonal. Then the adjoint of f is given by the conjugate-transpose - which will be equal to f by the previous statement. (TA)

Problem 1.2. Give an example of a real inner product space V and an operator $f : V \to V$ such that $(f - h)^2 + k^2$ is *not* invertible for some $h, k \in \mathbb{R}$ and $k > 0$.

Solution. Consider the matrix encoding of an operator $f : \mathbb{R}^2 \to \mathbb{R}^2$ equipped with the Euclidean norm as:

$$f \to A = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$

Let $h = 0$. Then we see

$$A^2 = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $A^2 + 1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus this operator satisfies the desired condition for $h = 0$ and $k = 1$. (TA)

Problem 1.3. Prove or give a counterexample: if there exists an orthonormal basis $(e_1, \ldots, e_n)$ such that an operator $f : V \to V$ satisfies $||f(e_j)|| = ||f^*(e_j)||$ then f is normal.

Solution. Consider the following matrix (as an operator in $\mathbb{R}^2$ with the Euclidean norm):

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

This matrix is normal if and only if it commutes with its adjoint. We test that:

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}$$

As we can see, the order of multiplication matters and therefore A isn't normal. But consider the action of A on the orthonormal basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$:

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$
$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

We see that $||Ae_i|| = ||A^T e_i||$ for $i = 1, 2$ and therefore the matrix satisfies the hypothesis of the problem. This means we have given a counterexample. (TA)

2 For submission to Davis Lazowski

Problem 2.1. Take V to be a finite-dimensional inner product space and $f\colon V \to V$ a linear operator on it. Show that λ is an eigenvalue of f exactly when $\overline{\lambda}$ is an eigenvalue of the adjoint map f^*. (Note: because f is *not* assumed normal, the associated eigenvectors may differ.)

Solution. Recall that f^* is the operator such that

$$\langle fu, w \rangle = \langle u, f^* w \rangle$$

As will be proved in problem 2.2.

Let $fv = \lambda v$.

Then

$$\langle fv, w \rangle = \langle \lambda v, w \rangle = \langle v, f^* w \rangle$$

Therefore, $\langle v, \overline{\lambda} w \rangle = \langle v, f^* w \rangle$.

Therefore, $\langle v, \overline{\lambda} w - f^* w \rangle = 0$. So $\overline{\lambda} w - f^* w \in \mathrm{span(v)}^\perp$ for every w. So $\dim \operatorname{im} (\overline{\lambda} - \mathrm{f}^*) \leq \mathrm{n} - 1$. Therefore, $\overline{\lambda} - f^*$ is not injective, so there is some $\tilde{w} \neq 0, \overline{\lambda}\tilde{w} - f^*\tilde{w} = 0$.

Therefore, $f^*\tilde{w} = \overline{\lambda}\tilde{w}$, as required. A completely symmetrical proof shows the opposite direction. (DL)

Problem 2.2. Suppose that $f\colon V \to W$ is a linear map between real inner product spaces. Recall the definition of the adjoint map from class:

$$\begin{tikzcd}
V^* \arrow[r, bend left, red] & V \arrow[l, bend left] \\
W^* \arrow[u, red, "\text{the dual map, } f^*"] \arrow[r, bend left] & W \arrow[l, bend left, red] \arrow[u, dotted, "\text{the adjoint map, } f^*"']
\end{tikzcd}$$

where, e.g., $W \to W^*$ is the isomorphism $w \mapsto \langle -, w \rangle$. Using this definition, check the claim

$$\langle f(v), w \rangle_W = \langle v, f^*(w) \rangle_V.$$

Solution. Denoting the dual map as f^*_{dual}, the adjoint as f^*_{adj}, and the functional: $\varphi_w(y) = \langle y, w \rangle_W$, so that $\varphi_w \in W^*$:

$$\begin{aligned}
&\langle f(v), w \rangle_W \\
&= \varphi_w(f(v)) \text{ (by definition of } \varphi_w\text{, above)} \\
&= f^*_{dual}(\varphi_w)(v) \text{ (by definition of the dual map)} \\
&= \langle v, f^*_{adj}(W) \rangle_V \text{ (by definition of the adjoint)}
\end{aligned}$$

To elucidate in words, define $\psi : V^* \to V$ to be the map which sends a functional $\gamma \in V^*$ to the associated vector $u \in V$. Recall that this map is defined such that $\gamma(v) = \langle v, u \rangle \, \forall v \in V$. In other words, $\gamma(v) = \langle v, \psi(\gamma) \rangle$. Also let $\varphi : W \to W^*$ the map which sends $w \to \varphi_w$.

- **Step one**: By definition of φ_w, we have that $\varphi_w(f(v)) = \langle f(v), w \rangle_W$, as required.
- **Step two**: The dual map is defined so that $\varphi_w \circ f = f^*_{dual}(\varphi_w)$. Apply this definition to $\varphi_w(f(v))$ to recover the result.
- **Step three**: By definition of the adjoint, $f^*_{adj} = \psi \circ f^*_{dual} \circ \varphi$. By definition of ψ,

$$\begin{aligned}
(f^*_{dual} \circ \varphi)(w)(v) &= \langle v, (\psi \circ f^*_{dual} \circ \varphi)(w) \rangle \\
&= \langle v, f^*_{adj}(w) \rangle
\end{aligned}$$

(DL)

Problem 2.3. Suppose $f\colon V \to V$ is self-adjoint. Suppose for $\lambda \in \mathbb{R}$ and $\varepsilon > 0$ there exists a $v \in V$ with $\|v\| = 1$ and

$$\|f(v) - \lambda \cdot v\| < \varepsilon.$$

Show that f has an eigenvalue λ' such that $|\lambda - \lambda'| < \varepsilon$.

Solution. $f - \lambda$ is self-adjoint again, so orthonormally diagonalisable. Let $e_1 ... e_n$ its eigenbasis, and $\gamma_1 ... \gamma_n$ is eigenvalues.

Let $||v|| = 1$. Then

$$\min\{||(f - \lambda)e_1||, ||(f - \lambda)e_2||...\} \leq ||(f - \lambda)v||$$

This is because v is a linear combination of the e_j, say $\sum \alpha_i e_i$, so that, with e_m the minimal eigenvector,

$$\begin{aligned}
&||(f - \lambda) \sum \alpha_i e_i|| \\
&= || \sum \gamma_i \alpha_i e_i|| \\
&= \sum |\gamma_i \alpha_i| \geq \sum |\gamma_m \alpha_i| \\
&\geq |\gamma_m|
\end{aligned}$$

Therefore, $|\gamma_m| \leq ||(f - \lambda)v|| < \varepsilon$. Since $|\gamma_m| = |\lambda' - \lambda|$, with λ' an eigenvalue of f, then done. (DL)

3 For submission to Handong Park

Problem 3.1. Suppose that V is a finite-dimensional real vector space, and suppose that $f\colon V \to V$ is a linear operator on V. Prove that V has a basis consisting of eigenvectors of f if and only if there is an inner product on V for which f is self-adjoint.

Solution. One direction here is simple - if f is self-adjoint for some inner product on V, then by the Real Spectral Theorem (as in Axler and class), we know that there exists not only a basis consisting of eigenvectors of f, but in particular, an orthonormal basis consisting of eigenvectors of f.
For the opposite direction, suppose that we have a basis of V that consists entirely of eigenvectors $e_1, ..., e_n$ of f. Then define the following inner product $\langle e_i, e_j \rangle$ for any $1 \le i \le j \le n$:

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

In addition, we will just define our inner product to have the rules that

$$\langle a e_i, e_j \rangle = a \langle e_i, e_j \rangle$$

and

$$\langle e_i + e_k, e_j \rangle = \langle e_i, e_j \rangle + \langle e_k, e_j \rangle$$

By definition of the inner product, we will then automatically have that this eigenvector basis is now orthonormal, since $||e_i|| = \sqrt{\langle e_i, e_i \rangle} = 1$ for each eigenvector e_i, and the inner product of any two different eigenvectors is 0. Since we have an orthonormal eigenvector basis, we'd then have that f is indeed self-adjoint for this inner product on V.
We need only check that this is indeed actually an inner product on V. We defined our inner product (by its rules) to satisfy linearity, so all we need now is symmetry and positive definiteness:

- Symmetric: $\langle e_i, e_j \rangle = \langle e_j, e_i \rangle = 0$ whenever $i \neq j$, and $\langle e_i, e_i \rangle = 1$ for all $1 \le i \le n$.
- Positive Definite: $\langle e_i, e_i \rangle \ge 0$ for all e_i.

These hold for the basis eigenvectors, so they will hold (by our linearity in the definition of the inner product) for any linear combinations as well, making this an inner product on V.
Intution-wise, we can also just recognize that once we define the inner product this way, we are essentially taking the dot product on coordinate vectors in $\mathbb{R}^n$ after we change basis from the standard basis of $\mathbb{R}^n$ to the new basis of eigenvectors $e_1, ..., e_n$ - and we know that the dot product on $\mathbb{R}^n$ works out to be a nice inner product on $\mathbb{R}^n$. (HP)

Problem 3.2. Suppose that V is a finite-dimensional inner product space, that $P\colon V \to V$ is a linear function, and that $P \circ P = P$.

1. Show that if $\operatorname{im} P \perp \ker P$, then there exists a subspace $U \le V$ such that $P = P_U$.
2. Show that if $\|Pv\| \le \|v\|$ for all $v \in V$, then there exists a subspace $U \le V$ such that $P = P_U$.
3. Show that P is self-adjoint if and only if there is a subspace $U \le V$ with $P = P_U$.

Solution.

1. For this part, just take $U = \operatorname{im} P$. Then, from a previous problem set, we have already proved that $\operatorname{im} P \oplus \ker P = V$ for projections P where $P = P \circ P$. We can then easily check that since we know that $\operatorname{im} P \perp \ker P$, P just ends up being precisely the orthogonal projection to $\operatorname{im} P$. We know that given any $v \in V$, $v = u + w$ for some $u \in \operatorname{im} P$ and $w \in \ker P$, but then $P(u + w) = P(u) + P(w) = P(u) = P_U(v)$ as hoped. All we have to show now is that $u - P(u) \perp U = \operatorname{im} P$. $P(u - P(u)) = P(u) - P(P(u)) = P(u) - P(u) = 0$, so $u - P(u) \in \ker P$. Since $\ker P \perp \operatorname{im} P = U$, we have that $v - P(v) = u - P(u) \perp U$, showing that P will not only be a projection ($P^2 = P$), but will be precisely the orthogonal projection onto $U = \operatorname{im} P$, so that $P = P_U$ if we choose $U = \operatorname{im} P$.
2. For this part, we again guess $U = \operatorname{im} P$, we again have $\operatorname{im} P \oplus \ker P = V$, and we would like to reduce to the first case by showing that $\|Pv\| \le \|v\|$ forces $\ker P = U^{\perp}$. Use the direct sum decomposition to

write $v \in V$ as $u + w$ for $u \in \operatorname{im} P$ and $w \in \ker P$. We then have:

$$\begin{aligned} \|P(v)\|^2 &\leq \|v\|^2 \\ \|u\|^2 &\leq \langle u, u\rangle + \langle u, w\rangle + \langle w, u\rangle + \langle w, w\rangle \\ 0 &\leq 2\operatorname{Re}\langle u, w\rangle + \|w\|^2. \end{aligned}$$

Our goal is to show that $\langle u, w\rangle$ is identically zero for all choices of $u \in U$ and $w \in \ker P$. Suppose that this weren't the case, and that we could find such u and w with $\langle u, w\rangle$ nonzero. We then modify u: if $\langle u, w\rangle$ is purely imaginary, multiply u by i so that it becomes real (so that $\langle u, w\rangle$ is now guaranteed to have a real component). If $2\operatorname{Re}\langle u, w\rangle$ is positive, multiply u by -1 so that $2\operatorname{Re}\langle u, w\rangle$ becomes negative. Finally, scale u by $\|w\|^2/\operatorname{Re}\langle u, w\rangle$ so that the last inequality above is violated. Since this conclusion is not possible, $\langle u, w\rangle$ must always have been identically zero.

3. For this part, we know that first, if P is self-adjoint, then we can take an orthonormal basis of eigenvectors of P. Then, the matrix representation of P on this orthonormal basis will be diagonal, since the matrix must be self-adjoint. By this matrix, we can then pick a basis for $\operatorname{im} P$ by picking the vectors e_j for which the j'th entry along the diagonal is not zero, and put the remaining orthonormal eigenvectors in a basis for $\ker P$. Then we automatically get that $\ker P \perp \operatorname{im} P$ by these disjoint bases whose vectors are all orthonormal to each other, and in part (1), we showed that this implies that $P = P_U$.
   Now suppose that $P = P_U$. We want to show that P is self-adjoint. To do so, we can just show that P_U is self-adjoint. For any vectors $v \in V$, we have $v = u + w$ where $u \in U$ and $w \in U^\perp$. Then we have for any $v_1, v_2 \in V$, by the properties of P_U being an orthogonal projection,

$$\langle P_U(v_1), v_2\rangle = \langle u_1, u_2 + w_2\rangle = \langle u_1, u_2\rangle = \langle u_1 + w_1, u_2\rangle = \langle P_U(v_2), v_1\rangle$$

   showing that P_U is self-adjoint. (HP)

Problem 3.3. Suppose that $f, g\colon V \to V$ are both self-adjoint. Show that $f \circ g$ is self-adjoint if and only if $f \circ g = g \circ f$.[1]

Solution. First, suppose f and g are both self-adjoint, and that $f \circ g$ is self-adjoint. Then we have

$$f \circ g = (f \circ g)^*$$

But we also know that

$$(f \circ g)^* = g^* \circ f^* = g \circ f$$

Thus

$$f \circ g = g \circ f$$

as we hoped to show.
Now, suppose that $f \circ g = g \circ f$, and that f and g are self-adjoint. Then

$$(f \circ g)^* = g^* \circ f^* = g \circ f$$

But since $g \circ f = f \circ g$, we must then have that

$$(f \circ g)^* = f \circ g$$

proving that $f \circ g$ is self-adjoint as well, concluding our proof. (HP)

[1] The functions f and g are said to "commute".

4 For submission to Rohil Prasad

Problem 4.1. Suppose that f is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of f. Show that

$$f^2 - 5f + 6 = 0$$

Show additionally that this conclusion fails if f is not self-adjoint.

Solution. Let the inner product space be V. Note that whether V is real or complex does not matter, since by Problem 1.1 if f has real eigenvalues and is normal then it is automatically self-adjoint.

Apply the Spectral Theorem. It follows that any $v \in V$ can be expressed as a linear combination $\lambda_2 v_2 + \lambda_3 v_3$, where v_2, v_3 are eigenvectors of f with eigenvalues 2 and 3 respectively.

Then

$$(f^2 - 5f + 6)(v) = 4\lambda_2 v_2 + 9\lambda_3 v_3 - 10\lambda_2 v_2 - 15\lambda_3 v_3 + 6\lambda_2 v_2 + 6\lambda_3 v_3 = 0$$

As an example of a map that is not self-adjoint but has eigenvalues 2 and 3 is the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ represented by the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

in the standard basis. We calculate $T^2 - 5T + 6$ to be

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

which is not the zero map. Therefore, we now just need to show that it has eigenvalues 2 and 3 and is not self-adjoint.

Applying T to the vector $v = (v_1, v_2, v_3)$ we note $T(v) = (2v_1, 3v_2 + 3v_3, 3v_3)$.

Equating this to $2v$, we find that we require $3v_3 = 2v_3$, so $v_3 = 0$. This implies $3v_2 = 2v_2$, so $v_2 = 0$ as well. Thus, the eigenspace with eigenvalue 2 is the span of $e_1 = (1, 0, 0)$.

Equating this to $3v$, we find similarly that $v_1 = 0$. Also, $3v_2 + 3v_3 = 3v_2$, so $v_3 = 0$ and therefore the eigenspace with eigenvalue 3 is the span of $e_2 = (0, 1, 0)$.

If $(2v_1, 3v_2 + 3v_3, 3v_3) = (\lambda v_1, \lambda v_2, \lambda v_3)$ for $\lambda \neq 2, 3$ then we would require v_1, v_3 equal to 0 immediately. However, this also implies $3v_2 = \lambda v_2$, so since $\lambda \neq 3$ we have $v_2 = 0$. Therefore, there cannot be any other eigenvalues other than 2 or 3.

This map is clearly not orthonormally diagonalizable, so by the spectral theorem it cannot be self-adjoint. (RP)

Problem 4.2. Consider the vector space $V = C[-\pi, \pi]$ of continuous functions with signature $[-\pi, \pi] \to \mathbb{R}$ which has an inner product specified by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$$

For an $n \in \mathbb{N}_{>0}$ construct a subspace

$$U_n = \operatorname{span}\{1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx\}$$

1. Define $D : U_n \to U_n$ by differentiation: $Df = f'$. Show that $D^* = -D$.
2. Is D normal? Is it self-adjoint? What about $D^{\circ 2}$?

Solution. 1. Note that by integration by parts, $\int gDf = fg - \int fDg$. Taking the integral from $-\pi$ to π, we find $\langle Df, g\rangle + \langle f, Dg\rangle = fg(\pi) - fg(-\pi)$.

However, all of the functions in U_n have period 2π, so if $f, g \in U_n$ then $fg(\pi) = fg(-\pi)$. It follows that $\langle Df, g\rangle = -\langle f, Dg\rangle = \langle f, -Dg\rangle$ and $D^* = -D$.

2. We have $DD^* = D \circ (-D) = -D^2 = (-D) \circ D = D^*D$, so D is normal. However, it is clear that D is not self-adjoint since $D^* = -D \neq D$.

On the other hand, D^2 is self-adjoint. Given functions $f, g \in U_n$ we can calculate by adjointness

$$\begin{aligned}\langle D^2 f, g\rangle &= \langle Df, -Dg\rangle \\ &= \langle f, (-D \circ -D)g\rangle \\ &= \langle f, D^2 g\rangle.\end{aligned}$$

(RP)

Problem 4.3. Recall that our proof of the real spectral theorem rests primarily on knowing that the perpendicular subspace of an invariant subspace is itself invariant. Rewrite our proof of the real spectral theorem to apply to a normal operator on a complex vector space.

Solution. Let our operator be a map $f : V \to V$ where V is a finite-dimensional complex vector space of dimension n.

Recall that every operator on a finite-dimensional complex vector space admits an eigenvector (Axler Theorem 5.10).

We will show the complex spectral theorem by inducting on the dimension n. In the case that $n = 1$, the theorem is trivial by the above assertion.

Now assume that the theorem holds for vector spaces of dimension less than n. Let $u \in V$ be an eigenvector of f, let $U \subset V$ be its span, and let $U^\perp$ be its orthogonal complement. Since $U \oplus U^\perp = V$ and U is spanned by an eigenvector of f, V is orthonormally diagonalizable if and only if $U^\perp$ is.

If $U^\perp$ is invariant under f, then since it has dimension $n-1$ we are done since we can apply the inductive hypothesis to the restricted operator $f|_{U^\perp}$. Pick $w \in U^\perp$. Showing $f(w) \in U^\perp$ is equivalent to showing for any $u \in U$ that $\langle f(w), u\rangle = 0$. Recall that since f is normal, and u is an eigenvector of f with eigenvalue λ, then u is also an eigenvector of f^* with eigenvalue $\overline{\lambda}$. Therefore, we have

$$\langle f(w), u\rangle = \langle w, f^*(u)\rangle = \langle w, \overline{\lambda} u\rangle = \lambda\langle w, u\rangle = 0$$

as desired. (RP)