Homework #7

Math 25a

Due: October 26, 2016

Guidelines:

- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one (though you should), a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!)

1 For submission to Thayer Anderson

Problem 1.1. Suppose that $f, g \colon K^3 \to K^3$ are two linear functions that each have eigenvalues 2, 6, and 7. Show that there exists a linear function $h \colon K^3 \to K^3$ satisfying $f = h \circ g \circ h^{-1}$.

Problem 1.2. A *norm* on V is a function $\| - \| \colon V \to \mathbb{R}_{\geq 0}$ satisfying

- $\|u\| = 0$ if and only if $u = 0$.
- $\|k \cdot u\| = |k| \cdot \|u\|$ for any scalar k.
- $\|u + v\| \leq \|u\| + \|v\|$.

In this problem, we will show that when a norm arises from an inner product by $\|v\| = \sqrt{\langle v, v\rangle}$, we can recover the inner product from the norm.

1. Suppose that V is a real inner product space. Show that

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}.$$

2. Suppose that V is a complex inner product space. Show that

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4}.$$

2 For submission to Davis Lazowski

Problem 2.1. Suppose that $S\colon V \to V$ is a linear operator on an inner product space V. Define a new pairing by

$$\langle u, v\rangle_S = \langle Su, Sv\rangle.$$

1. Suppose that S is injective. Show that this new pairing is also an inner product on V.

2. Suppose that S *fails* to be injective. Show that this same pairing is *not* an inner product on V.

Problem 2.2. Suppose V is a finite–dimensional real vector space, and suppose $\langle -, -\rangle_1$ and $\langle -, -\rangle_2$ are two inner products on V.

1. Show that there exists a number $c > 0$ with $\|v\|_1 \le c\|v\|_2$.

2. Suppose further that $\langle v, w\rangle_1 = 0$ if and only if $\langle v, w\rangle_2 = 0$. Show that there is a number $c > 0$ such that $\langle -, -\rangle_1 = c \cdot \langle -, -\rangle_2$.

3 For submission to Handong Park

Problem 3.1. What happens if Gram–Schmidt is applied to a list of vectors that is not linearly independent?

Problem 3.2. Suppose V is a finite–dimensional complex vector space, and suppose $f\colon V \to V$ is a linear function whose eigenvalues are all of absolute value less than 1. For any $\varepsilon > 0$, show there exists a positive integer m with $\|T^m v\| < \varepsilon\|v\|$ for every $v \in V$. (Hint: you could begin with an upper-triangular presentation of f.)

4 For submission to Rohil Prasad

Problem 4.1.

1. On $P_2(\mathbb{R})$, consider the inner product given by

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx.$$

   Apply the Gram–Schmidt procedure to the basis $(1, x, x^2)$ to produce an orthonormal basis of $P_2(\mathbb{R})$.

2. Find a polynomial $q \in P_2(\mathbb{R})$ such that for every $p \in P_2(\mathbb{R})$,

$$p\left(\frac{1}{2}\right) = \langle p, q\rangle$$

   under the same inner product.

Problem 4.2. The Fibonacci sequence $F_1, F_2, \ldots$ is defined by

$$F_1 = 1, \qquad F_2 = 1, \qquad F_n = F_{n-2} + F_{n-1}.$$

We also define a linear function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (y, x + y)$.

1. Show that $T^n(0, 1) = (F_n, F_{n+1})$.[1]

2. Find the eigenvalues of T.

3. Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.

[1] This used to read (F_n, F_{n-1}), which was a typo. Sorry!

4. Use the solution to the previous part to compute $T^n(0,1)$ in closed form.

5. Conclude more lazily that the n^{th} Fibonacci number F_n is the nearest integer to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$