Homework #6

Math 25a

Due: October 19, 2016

Guidelines:

- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one (though you should), a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!)

1 For submission to Thayer Anderson

Problem 1.1. Let V be a finite-dimensional real vector space, $f\colon V \to V$ a linear map, and $\lambda \in \mathbb{R}$ some real number. Show that there exists a second real number $\alpha \in \mathbb{R}$ with $|\lambda - \alpha| < \frac{1}{1000}$ such that $f - \alpha$ is invertible.

Problem 1.2. Let A be an $(n \times n)$–matrix presenting a linear function $\mathbb{R}^n \to \mathbb{R}^n$.

1. Suppose that the sum of the entries in each *row* of A equals 1. Show that 1 is an eigenvalue of A.
2. Suppose that the sum of the entries in each *column* of A equals 1. Show that 1 is an eigenvalue of A.

Problem 1.3. Let V be finite dimensional and let be $f\colon V \to V$ a linear function. Suppose that $v \in V$ is a nonzero vector, and suppose that p is a nonzero polynomial with $p(f)(v) = 0$, and suppose that there are *no* polynomials of degree less than that of p which have this property. Show that every zero of p is an eigenvalue of f.

2 For submission to Davis Lazowski

Problem 2.1. Suppose $f\colon V \to V$ is invertible. Show that λ is an eigenvalue of f if and only if λ^{-1} is an eigenvalue of f^{-1}, and show that v is an eigenvector of f if and only if it is also an eigenvector of f^{-1}.

Problem 2.2. Suppose $f\colon V \to V$ is a linear transformation with $\dim \operatorname{im} f = k$. Show that f has at most $(k+1)$ distinct eigenvalues.

Problem 2.3. Suppose V is a *complex* vector space, $f\colon V \to V$ a linear function, and p a complex polynomial. Show that $\alpha \in \mathbb{C}$ is an eigenvalue of $p(f)$ if and only if $\alpha = p(\lambda)$ for some eigenvalue λ of f. Then, show that this result fails if V is merely assumed to be a real vector space and p a real polynomial.

3 For submission to Handong Park

Problem 3.1. Let $p\colon V \to V$ satisfy $p \circ p = p$. Show that $V = \ker p \oplus \operatorname{im} p$.

Problem 3.2. Suppose that $f\colon V \to V$ is a linear operator with $f \circ f = \mathrm{id}$, and suppose that -1 is *not* an eigenvalue of f. Show that $f = \mathrm{id}$.

Problem 3.3.

1. Suppose that a subspace $U \leq V$ is invariant under a linear function $f\colon V \to V$. Show that U is also invariant under $p(f)$, where p is any polynomial.

2. Now suppose V is a complex vector space with dimension $1 < \dim V < \infty$. Show that for any particular linear map $f\colon V \to V$, there is a *proper* subspace

$$\{p(f) \mid p \text{ a polynomial}\} < \mathcal{L}(V, V).$$

4 For submission to Rohil Prasad

Problem 4.1. Let $f\colon V \to V$ be a linear operator. Prove that $f/\ker f$ is injective if and only if

$$(\ker f) \cap (\operatorname{im} f) = 0.$$

Problem 4.2. Suppose that $\lambda_1, \ldots, \lambda_n$ is a list of distinct real numbers. Show that $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ is a list of linearly independent functions $\mathbb{R} \to \mathbb{R}$. (Hint: find a linear operator on the space of functions $\mathbb{R} \to \mathbb{R}$ for which these are eigenvectors of distinct eigenvalues.)

Problem 4.3. Let V be an arbitrary vector space and let $f\colon V \to V$ be a linear function. Consider the following three situations:

1. Every nonzero vector is an eigenvector of f.

2. The vector space V is finite dimensional of dimension n, and every subspace $U \leq V$ with $\dim U = n-1$ is invariant under f.

3. The vector space V is finite dimensional of dimension $n \geq 3$, and every subspace $U \leq V$ with $\dim U = 2$ is invariant under f.

In each case, show that f is a scalar multiple of the identity operator.