Homework #5

Math 25a

Due: October 12, 2016

Guidelines:

- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one (though you should), a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!)

1 For submission to Thayer Anderson

Problem 1.1. Suppose that both V and W are finite dimensional. Show that the assignment

$$\begin{aligned} \mathcal{L}(V, W) &\to \mathcal{L}(W^*, V^*) \\ \varphi &\mapsto \varphi^* \end{aligned}$$

is an isomorphism of vector spaces.

Problem 1.2. Consider a linear map $f\colon V \to V$ and an isomorphism $\varphi\colon V \xrightarrow{\simeq} W$.

1. Prove that f and $\varphi \circ f \circ \varphi^{-1}\colon W \to W$ have the same eigenvalues.
2. What is the relationship between eigenvectors for f and eigenvectors for $\varphi \circ f \circ \varphi^{-1}$?

Problem 1.3. Show that a degree m polynomial p has m distinct zeroes exactly if p and its derivative p' have no zeroes in common.

2 For submission to Davis Lazowski

Problem 2.1. Suppose p is a complex polynomial. Show that $q = p \cdot \overline{p}$ is a polynomial with real coefficients.

Problem 2.2. Consider a complex vector space V, a map $f\colon V \to V$, and a basis of V in which f is expressed by a matrix M with all real entries. Show that if λ is an eigenvalue of f, then so is $\overline{\lambda}$.

Problem 2.3. Suppose that V is finite-dimensional and let $f, g\colon V \to V$ be linear functions. Show that $f \circ g$ and $g \circ f$ have the same eigenvalues.

3 For submission to Handong Park

Problem 3.1. Suppose that $U_1, \ldots, U_n \leq V$ are invariant subspaces under an operator $f\colon V \to V$. Show that their intersection $U_1 \cap \cdots \cap U_n$ and their subspace sum $U_1 + \cdots + U_n$ are invariant under f as well.

Problem 3.2. Find all eigenvalues and eigenvectors of the backward shift operator $f\colon K^\infty \to K^\infty$ defined by

$$f(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Problem 3.3. Suppose f is a nonzero polynomial, and let U be the subspace of all polynomials P defined by

$$U = \{f \cdot g \mid g \text{ a polynomial}\}.$$

Show that $\dim P/U = \deg f$, and exhibit a basis of P/U.

Problem 3.4. Show that every real polynomial of odd degree has a zero.

4 For submission to Rohil Prasad

Problem 4.1. Show that the dual basis of $(1, x, x^2, \ldots, x^n)$ of P_n is $\varphi_0, \ldots, \varphi_n$ defined by

$$\varphi_j(f) = \frac{f^{(j)}(0)}{j!}.$$

Problem 4.2. Consider the differentiation operator on the vector space P of all polynomials:

$$\frac{d}{dx}\colon P \to P.$$

Calculate all the eigenvectors and eigenvalues of $\frac{d}{dx}$.

Problem 4.3. Let p be a complex polynomial of degree m, and suppose that there are distinct $x_0, \ldots, x_m \in \mathbb{R}$ with $p(x_j) \in \mathbb{R}$ for all j. Prove that p is actually a real polynomial.