Homework #5 Solutions

Thayer Anderson, Davis Lazowski, Handong Park, Rohil Prasad
Eric Peterson

1 For submission to Thayer Anderson

Problem 1.1. Suppose that both V and W are finite dimensional. Show that the assignment

$$\mathcal{L}(V,W) \to \mathcal{L}(W^*, V^*)$$
$$\varphi \mapsto \varphi^*$$

is an isomorphism of vector spaces.

Solution. Let the given assignment be called Φ. We wish to show that Φ is surjective, injective, and linear. First we prove linearity. Suppose $f, g \in \mathcal{L}(V,W)$ and $c \in K$. Then

$$\Phi(f + cg) = (f + cg)^*$$

I claim that $(f + cg)^* = f^* + cg^*$. To prove this, I will consider the following quantity:

$$(f + cg)^*(\lambda)(v)$$

for $\lambda \in W^*$ and $v \in V$. By the definition of the dual map this is equal to

$$\lambda((f + cg)(v)) = \lambda(f(v) + cg(v))$$
$$= \lambda(f(v)) + c\lambda(g(v)) = f^*(\lambda)(v) + cg^*(\lambda)(v)$$

This completes the proof of linearity. To prove that it is an isomorphism, I first prove that it is injective. Suppose $\Phi(f) = f^* = 0$. Then take a basis $w_1, \dots, w_n$ for W. Then

$$0 = f^*(w_i^*)(v) = w_i^*(f(v))$$

for arbitrary $v \in V$. It follows that each dual basis element is mapped to 0 and therefore $f^* = 0$. This completes the proof of injectivity. Moreover, the dimensions of the vector spaces are equal so the map is necessarily an isomorphism. (TA)

Problem 1.2. Consider a linear map $f\colon V \to V$ and an isomorphism $\varphi\colon V \simeq W$.

1. Prove that f and $\varphi \circ f \circ \varphi^{-1}\colon W \to W$ have the same eigenvalues.
2. What is the relationship between eigenvectors for f and eigenvectors for $\varphi \circ f \circ \varphi^{-1}$.

Solution. I will prove these parts together, as the one falls out from the other. Suppose $v \in V$ is an eigenvector of f with eigenvalue λ. Then consider

$$\varphi \circ f \circ \varphi^{-1}(\varphi(v)) = \varphi \circ f(v) = \lambda\varphi(v)$$

and thus $\varphi(v)$ is an eigenvector of $\varphi \circ f \circ \varphi^{-1}$ with eigenvalue λ. This logic has a slight flaw if $\varphi(v) = 0$, but in that case $v = 0$ by injectivity of φ and thus v was not an eigenvector.

This tells us that the eigenvalues of f are a subset of the eigenvalues of $\varphi \circ f \circ \varphi^{-1}$. Similarly, suppose w is an eigenvector of $\varphi \circ f \circ \varphi^{-1}$ with eigenvalue λ. Then $w = \varphi(v)$ for some fixed $v \in V$ (by surjectivity of φ) and we see that

$$\begin{aligned} \lambda w = \varphi \circ f \circ \varphi^{-1}\varphi(v) &= \varphi \circ f(v) \\ \Rightarrow \lambda\varphi^{-1}(w) &= f(v) \\ \Rightarrow \lambda v &= f(v) \end{aligned}$$

This completes the proof that the eigenvalues are shared are gives the second relationship between eigenvectors. (TA)

Problem 1.3. Show that a degree m polynomial p has m distinct zeroes exactly if p and its derivative p' have no zeroes in common.

Solution. Suppose p and p' have a root in common. Then we may write

$$p(x) = (x-a)r_1(x)$$
$$p'(x) = (x-a)r_2(x)$$

for some polynomials r_1 and r_2. Then we calculate

$$p'(x) = (x-a)r_1'(x) + r_1(x) = (x-a)r_2(x)$$

From this form we see that $(x-a)$ divides r_1 and therefore a is at least a double root of p.

For the other direction, suppose that p has a double root, that is

$$p = (x-a)^2 r(x)$$

for some polynomial r. Then we calculate the derivative:

$$p' = 2(x-a)r(x) + (x-a)^2 r'(x)$$

and thus a is a root of p' and p has a root in common with p'. This completes the proof. (TA)

2 For submission to Davis Lazowski

Problem 2.1. Suppose p is a complex polynomial. Show that $q = p \cdot \overline{p}$ is a polynomial with real coefficients.

Solution. Certainly, $q(x) = p(x)\overline{p}(x)$ is a real number for all real x. In particular, $q^{(j)}(0) \in \mathbb{R} \forall j$.

Therefore $\varphi_j(q) : \varphi_j(q) = \frac{q^{(j)(0)}}{j!} \in \mathcal{L}(P^n, \mathbb{R})$, where $n = \deg q$. In particular, by problem 4.1 this means that the dual vector $q^* \in (P^n)^*$ where $(P^n)^*$ is over the field $\mathbb{R}$ because it is in the span of the basis vectors.

Therefore, $q \in P^n$ as a vector space over the reals. Therefore, $q = \sum_{j=1}^n \lambda_j x^j$, where $\lambda_j \in \mathbb{R}$, therefore done. (DL)

Problem 2.2. Consider a complex vector space V, a map $f\colon V \to V$, and a basis of V in which f is expressed by a matrix M with all real entries. Show that if λ is an eigenvalue of f, then so is $\overline{\lambda}$.

Solution. Let $\lambda w = fw$. In matrix form, with the basis V, $\lambda(\sum_{j=1}^n \alpha_j v_j) = A(\sum_{j=1}^n \alpha_j v_j)$. Since A has all real entries, $A = \overline{A}$.

Therefore

$$\begin{aligned}
\overline{\lambda(\sum_{j=1}^{n}\alpha_j v_j)} &= \overline{A(\sum_{j=1}^{n}\alpha_j v_j)} \\
\implies \overline{\lambda}(\sum_{j=1}^{n}\overline{\alpha_j} v_j) &= \overline{A}(\sum_{j=1}^{n}\overline{\alpha_j} v_j) \\
\implies \overline{\lambda}(\sum_{j=1}^{n}\overline{\alpha_j} v_j) &= A(\sum_{j=1}^{n}\overline{\alpha_j} v_j) \\
\implies \overline{\lambda}(\sum_{j=1}^{n}\overline{\alpha_j} v_j) &= f(\sum_{j=1}^{n}\overline{\alpha_j} v_j)
\end{aligned}$$

(DL)

Problem 2.3. Suppose that V is finite-dimensional and let $f, g\colon V \to V$ be linear functions. Show that $f \circ g$ and $g \circ f$ have the same eigenvalues.

Solution. Let $[f \circ g]w = \lambda w$. Then $g \circ [f \circ g]w = g(\lambda w) = \lambda g(w)$. So that, by associativity of function composition, $g \circ f(gw) = \lambda gw$.

Therefore gw is an eigenvector of $g \circ f$, with same eigenvalue. Symmetrically

$$\begin{aligned}
[g \circ f]v &= \alpha v \\
\implies f[g \circ f]v &= f(\alpha v) \\
\implies [f \circ g](fv) &= \alpha(fv)
\end{aligned}$$

So that fv is an eigenvector of $f \circ g$ with eigenvalue also α. (DL)

3 For submission to Handong Park

Problem 3.1. Suppose that $U_1, \ldots, U_n \leq V$ are invariant subspaces under an operator $f\colon V \to V$. Show that their intersection $U_1 \cap \cdots \cap U_n$ and their subspace sum $U_1 + \cdots + U_n$ are invariant under f as well.

Solution. We prove each assertion in turn:

1. A vector $v \in U_1 \cap \cdots \cap U_n$ is simultaneously a member of each U_j. It follows that $f(v)$ is simultaneously a member of each U_j, since each U_j is invariant under f. This is the same as saying $f(v) \in U_1 \cap \cdots \cap U_n$, so that $U_1 \cap \cdots \cap U_n$ is invariant under f as well.

2. A vector $v \in U_1 + \cdots + U_n$ admits (possibly nonunique) expression as $v = u_1 + \cdots + u_n$ for $u_j \in U_j$. Applying f to this decomposition gives

$$f(v) = f(u_1 + \cdots + u_n) = f(u_1) + \cdots + f(u_n) = u_1' + \cdots + u_n',$$

where we have used that $f(u_j) = u_j'$ is again a member of U_j. From this, it follows that $f(v) \in U_1 + \cdots + U_n$, so that $U_1 + \cdots + U_n$ is invariant under f. (ECP)

Problem 3.2. Find all eigenvalues and eigenvectors of the backward shift operator $f\colon K^\infty \to K^\infty$ defined by

$$f(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Solution. The eigenvector equation for this operator is

$$\lambda \cdot (x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Expanding this into a system of equations, we have

$$\lambda x_1 = x_2, \qquad \lambda x_2 = x_3, \qquad \lambda x_3 = x_4, \qquad \ldots.$$

We see that for any choice of x_1 and λ, the rest of the x_j are uniquely determined. It follows that each eigenspace $E(k)$ is 1–dimensional, for *any* choice of $k \in K$. (ECP)

Problem 3.3. Suppose f is a nonzero polynomial, and let U be the subspace of all polynomials P defined by

$$U = \{f \cdot g \mid g \text{ a polynomial}\}.$$

Show that $\dim P/U = \deg f$, and exhibit a basis of P/U.

Solution. Write

$$f = a_n x^n + \cdots + a_1 x + a_0$$

for $a_n \neq 0$, and write $\pi\colon P \to P/U$ for the quotient. The key observation is that, after projection along π, we have

$$\begin{aligned} \pi(a_n x^n + \cdots + a_1 x + a_0) &= \pi(0) \\ \pi(a_n x^n) + \pi(a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) &= 0 \\ \pi(a_n x^n) &= -\pi(a_{n-1}x^{n-1} + \cdots + a_1 x + a_0). \end{aligned}$$

In fact, $f(x) \cdot x^m$ is also a member of U, so

$$\pi(a_n x^{n+m}) = -\pi(a_{n-1}x^{n+m-1} + \cdots + a_1 x^{m+1} + a_0 x^m).$$

Induction shows that any polynomial of degree at least n can be rewritten as a polynomial of degree at most $n-1$, so that $(\pi(1), \pi(x), \ldots, \pi(x^{n-1}))$ forms a spanning set of P/U. If there were a linear dependence among these elements, then we would have

$$\begin{aligned} k_0\pi(1) + k_1\pi(x) + \cdots + k_{n-1}\pi(x^{n-1}) &= 0 \\ \pi(k_0 + k_1 x + \cdots + k_{n-1}x^{n-1}) &= 0, \end{aligned}$$

so that $k_0 + k_1 x + \cdots + k_{n-1}x^{n-1} \in \ker \pi = U$. This cannot be: because every member of U is formed as $f \cdot g$, the degree of every member of U satisfies

$$\deg(f \cdot g) = \deg f + \deg g = n + \deg g \geq n.$$

As this list spans and is linearly independent, it forms a basis. The length of the list shows

$$\dim P/U = n. \qquad \text{(ECP)}$$

Solution. Another way in which we can understand the proof and the result is to think of P/U in this case as an affine space given by the set of all possible polynomial remainders $r(x)$ when we divide any possible polynomial in P by the fixed polynomial $f(x)$. What our proof from above aims to do is to formalize the idea that given any polynomial $p(x)$, it can be written as (via long division of polynomials):

$$p(x) = r(x) + f(x) \cdot g(x), \text{ where } f(x) \cdot g(x) \in U$$

In other words, whenever we have a polynomial that is of degree n or greater, it can be "divided" into a multiple of f by some polynomial g added to some remainder polynomial r of degree strictly less than the degree of f. In P/U, we only care about the remainder $r(x)$ and not the portion that is a multiple of $f(x) \in U$, since any multiples of $f(x)$ provide no new unique polynomials when we mod out by U. So in order to have a basis for P/U, all we need to do is have a basis that generates the possible remainders $r(x)$ of degree $n-1$ down to 0, and $1, x, x^2, ..., x^{n-1}$ is one such simple basis, which we can demonstrate to be both linearly independent and generating.

(HP)

Problem 3.4. Show that every real polynomial of odd degree has a zero.

Solution. Axler assures us that every real polynomial factors essentially uniquely as a product of a scalar, some linear factors, and some irreducible quadratic factors. If the factorization of our real polynomial f consisted solely of quadratic factors, its expansion would be of even degree. Instead, we know that f has odd degree, so that there must be at least one linear factor, say $(x - b)$. It then follows that $f(b) = 0$. (ECP)

Solution. Alternatively, we showed in class that eventually the leading term of a polynomial dominates the rest of the expression. It follows that for large positive values, the polynomial f takes the sign of its leading coefficient a_n, and for large negative values, the polynomial f takes the sign of $-a_n$. Since $a_n \neq 0$, these are opposite signs, and the Intermediate Value Theorem guarantees the existence of a zero. (ECP)

4 For submission to Rohil Prasad

Problem 4.1. Show that the dual basis of $(1, x, x^2, \ldots, x^n)$ of P_n is $\varphi_0, \ldots, \varphi_n$ defined by

$$\varphi_j(f) = \frac{f^{(j)}(0)}{j!}$$

Solution. It suffices to show that $\varphi_j(x^j) = 1$ and $\varphi_j(x^i) = 0$ for every $i \neq j$.

If $i < j$, then $(x^i)^{(j)} = 0$ and so $\varphi_j(x^i) = 0$.

If $i = j$, then $(x^j)^{(j)} = j!$ and so $\varphi_j(x^j) = j!/j! = 1$.

If $i > j$, then $(x^i)^{(j)} = i(i-1)\ldots(i-j+1)x^{i-j}$, which is equal to 0 when evaluated at 0. Therefore, $\varphi_j(x^j) = 0$ as well. (RP)

Problem 4.2. Consider the differentiation operator on the vector space P of all polynomials:

$$\frac{d}{dx} : P \to P$$

Calculate all the eigenvectors and eigenvalues of P.

Solution. We will show that the only eigenvalue is 0 with corresponding eigenvector 1 (or any $\lambda \in K$).

The fact that these are an eigenvalue/eigenvector is immediate by the definition of differentiation.

Now we will show no polynomial of degree $d \geq 1$ can be an eigenvector. Assume for the sake of contradiction that $p(x) = \sum_{i=0}^{d} c_i x^i$ with $c_d \neq 0$ is an eigenvector of the differentiation operator with eigenvalue λ. By definition, $dp/dx = \sum_{i=0}^{d-1}(i+1)c_{i+1}x^i$.

Comparing the degree d coefficients of λp and dp/dx, we find that $\lambda c_d = 0$. Since $c_d \neq 0$, we must have $\lambda = 0$.

However, if $\lambda = 0$, then $dp/dx = \lambda p = 0$. The degree $d - 1$ coefficient of dp/dx is $dc_d \neq 0$, so we arrive at a contradiction and p is not an eigenvector of the differentiation operator. (RP)

Problem 4.3. Let p be a complex polynomial of degree m and suppose that there are distinct $x_0, \ldots, x_m \in \mathbb{R}$ with $p(x_j) \in \mathbb{R}$ for all j. Prove that p is actually a real polynomial.

Solution. We will prove this by induction on m.

In this base case, let $m = 0$. Then p is a constant $c \in \mathbb{C}$, so if $p(x_0) \in \mathbb{R}$ then we must have $c = p(x_0) \in \mathbb{R}$.

Now assume that this holds for polynomials of degree $m - 1$. Let p be a complex polynomial of degree m such that $x_0, \ldots, x_m \in \mathbb{R}$ satisfy $p(x_j) \in \mathbb{R}$ for all j.

Since $p(x_m) \in \mathbb{R}$, we have p has real coefficients if and only if $p - p(x_m)$ has real coefficients. By definition, $p - p(x_m)$ has x_m as a root, so it factors as a product $(x - x_m)q$, where q is a complex polynomial of degree $m - 1$.

Plugging in x_j for $j < m$, we find that $(x_j - x_m)q(x_j) \in \mathbb{R}$. Since $x_j - x_m \in \mathbb{R}$, it follows that $q(x_j) \in \mathbb{R}$. Since q has degree $m - 1$ and $x_0, \ldots, x_{m-1}$ satisfy $q(x_j) \in \mathbb{R}$ for every j, by our inductive hypothesis q is real and therefore $p - p(x_m)$ is real, which implies p is real. (RP)