# Homework #4

Math 25a

Due: October 5, 2016

Guidelines:

- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one (though you should), a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!)

## 1 For submission to Thayer Anderson

**Problem 1.1.** Suppose that $V$ is a vector space over a field of scalars $K$ with $\dim V = 1$. Show that each element $f \in \mathcal{L}(V, V)$ has the form $f(v) = \lambda_f \cdot v$ for some scalar $\lambda_f$ (which depends on $f$ but not on $v$). Conclude from this that there is an isomorphism of vector spaces

$$\mathcal{L}(V, V) \xrightarrow{\cong} K$$

which *does not depend* upon choosing a basis for $V$.

**Problem 1.2.** Suppose that $f\colon V \to W$ and $g\colon W \to X$ are linear maps.

1. Show that $\dim \ker(g \circ f) \leq \dim \ker f + \dim \ker g$.
2. Show that $\dim \operatorname{im}(g \circ f) \leq \min\{\dim \operatorname{im} f, \dim \operatorname{im} g\}$.

**Problem 1.3.** Let $f, g\colon V \to W$ be two linear maps.

1. Suppose that $W$ is finite dimensional. Show that $\ker f \subseteq \ker g$ if and only if there exists a third linear map $h\colon W \to W$ satisfying $g = h \circ f$.
2. Suppose that $V$ is finite dimensional. Show that $\operatorname{im} f \subseteq \operatorname{im} g$ if and only if there exists a third linear map $h\colon V \to V$ satisfying $f = g \circ h$.

## 2 For submission to Davis Lazowski

**Problem 2.1.** Actually check the distributivity of linear maps. Suppose you are given linear maps

$$f, g\colon V \to W$$

and a third linear map

$$h\colon W \to X.$$

Demonstrate $h(f + g) = (hf) + (hg)$.

**Problem 2.2.** Let $f\colon V \to W$ be a linear map and suppose that $V$ is finite dimensional. Show that there exists a subspace $U \leq V$ with $U \oplus \ker f = V$ and $f(U) = \operatorname{im} V$.

**Problem 2.3.** Suppose that $f\colon V \to W$ is a linear function of vector spaces $V$ and $W$ over a scalar field $K$, and let $(w_1, \ldots, w_n)$ be a basis for $\operatorname{im} f$. Show that there exist $\varphi_1, \ldots, \varphi_n \in \mathcal{L}(V, K)$ such that

$$f(v) = \varphi_1(v) \cdot w_1 + \cdots + \varphi_n(v) \cdot w_n.$$

**Problem 2.4.** Prove that the intersection of any finite collection of affine subsets of $V$ is either the empty set or yet another affine subset.

## 3 For submission to Handong Park

Your task is to reinvent Gaussian elimination, since this topic is not covered in the book.

**Problem 3.1.** [1] Gaussian elimination involves three *elementary row operations* performed on the entries of a matrix:

- Swap the $j^{\text{th}}$ and $k^{\text{th}}$ rows.
- For indices $j \neq k$ and some scalar $c$, take the $k^{\text{th}}$ row, scale all its entries by $c$, and add the result to the $j^{\text{th}}$ row.
- For an index $j$ and a *nonzero* scalar $c$, scale the $j^{\text{th}}$ row by $c$.

Our first goal is to understand some features of these row operations.

1. Each of these operations can be encoded by matrices $S(j, k)$, $A(j, k, c)$, and $M(j, c)$ so that $S(j, k) \cdot X$, $A(j, k, c) \cdot X$, and $M(j, c) \cdot X$ are the respective results of the operations applied to a matrix $X$. These matrices $S(j, k)$, $A(j, k, c)$, and $M(j, c)$ are called *elementary matrices*. Find descriptions of the elementary matrices. (This could mean formulas, or English descriptions, or. . . )
2. Check that the elementary matrices are all invertible.
3. You can think of $S(j, k)$, $A(j, k, c)$, and $M(j, c)$ as describing a change of basis. If $X$ encodes the behavior of a linear map $f\colon V \to W$ on a basis $(w_1, \ldots, w_n)$ of $W$, then in what bases do $S(j, k) \cdot X$, $A(j, k, c) \cdot X$, and $M(j, c) \cdot X$ encode $f$?

**Problem 3.2.** A matrix $X$ is said to be *upper-triangular* when the entries satisfy $X_{ij} = 0$ whenever $i > j$. (When written as a block of numbers, all the entries below the main diagonal are zero.) Describe an algorithm which modifies a matrix to be upper-triangular using the elementary row operations. (Be sure to argue that your algorithm actually succeeds at this goal.) (Hint: work one column at a time.)

[1] This problem originally flipped left- and right-multiplication. Hopefully it's straight now.

**Problem 3.3.**

1. Problem 3.2 can be used to calculate the inverses of matrices. Suppose that your algorithm row-reduces a matrix $X$ to the identity matrix, i.e.,

$$E_n \cdot \dots \cdot E_2 \cdot E_1 \cdot X = I$$

for some sequence of elementary matrices $(E_j)$. Supposing further that $X$ is invertible, i.e.,

$$X \cdot X^{-1} = I$$

for other some matrix $X^{-1}$, solve for $X^{-1}$ in terms of the elementary matrices $(E_j)$.

2. Use your algorithm to calculate the matrix inverse of

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

(If necessary, modify your algorithm above to handle this case — that is, make sure it gives you the identity matrix rather than merely an upper triangular matrix.)

**Problem 3.4.**

1. Describe the effect of *right-multiplying* some matrix $X$ by an elementary matrix.
2. Using both the original elementary row operations and the operations uncovered in the previous part, devise a variation on your answer to Problem 3.2 that gives an algorithm that rewrites *any* matrix as a *diagonal* matrix. (Again, be sure to argue that your algorithm actually succeeds at this goal.)
3. Conclude that for any linear map $f\colon V \to W$ between finite dimensional vector spaces, we can find bases of $V$ and $W$ such that the resulting matrix $X$ expressing $f$ is *diagonal*.
4. Conclude rank-nullity from this form $X$ for $f$:

$$\dim V = \dim \operatorname{im} f + \dim \ker f.$$

# 4 For submission to Rohil Prasad

**Problem 4.1.** Suppose that $x, y \in V$ are vectors in a vector space $V$ and $U, W \le V$ are subspaces of $V$, altogether satisfying the relation

$$x + U = y + W.$$

Show that $U = W$.

**Problem 4.2.** Prove that a nonempty subset $A \subseteq V$ of a vector space $V$ is an affine subset if and only if for all $v, w \in A$ and all $\lambda \in K$ it is also the case that

$$\lambda \cdot v + (1 - \lambda) \cdot w \in A.$$

(Side remark: for $K = \mathbb{R}$ and $0 \le \lambda \le 1$, this property is called *convexity*.)

**Problem 4.3.** Suppose that $U \le V$ is a subspace such that the quotient space $V/U$ is finite dimensional. Show that $V$ is isomorphic to $U \times (V/U)$.