Homework #3

Math 25a

Due: September 28, 2016

Guidelines:

- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one (though you should), a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.
- For the first few assignments, we would like you to keep track of how long it takes you to complete the (whole) assignment and to notate that somewhere on your submission, so that we can calibrate the difficulty of the homeworks.

1 For submission to Thayer Anderson

Problem 1.1. Let V be the vector space of polynomials of degree at most 3. Prove or disprove that there exists a basis of V consisting of polynomials, none of which are of degree 3.

Problem 1.2 (Follow-up to problem 3.2 from last time)**.** Exhibit an example of a vector space V with non-equal subspaces U_1, U_2, and U_3 such that

$$U_1 \oplus U_3 = U_2 \oplus U_3.$$

Problem 1.3.

1. Under what conditions on the scalars $a, b \in \mathbb{C}$ are the vectors

$$\begin{pmatrix} 1 \\ a \end{pmatrix}, \begin{pmatrix} 1 \\ b \end{pmatrix}$$

a linearly dependent set in $\mathbb{C}^2$?

2. Under what conditions on the scalars $a, b, c \in \mathbb{C}$ are the vectors

$$\begin{pmatrix} 1 \\ a \\ a^2 \end{pmatrix}, \begin{pmatrix} 1 \\ b \\ b^2 \end{pmatrix}, \begin{pmatrix} 1 \\ c \\ c^2 \end{pmatrix}$$

a linearly dependent set in $\mathbb{C}^3$? (Come up with conditions analogous to those in the previous part.)

3. State and prove an analogous condition on sequences of n scalars forming sequences of n vectors in $\mathbb{C}^n$.

2 For submission to Davis Lazowski

Problem 2.1. For two subspaces U and W of a vector space V, show that if every vector in V belongs to either U or W (or both) then it must be the case that $U = V$ or $W = V$ (or both).

Problem 2.2. Suppose that $v_1, \ldots, v_m \in V$ form a linearly independent set and let $w \in V$ be another vector. Show that

$$\dim\left(\operatorname{span}\{v_1 + w, \ldots, v_m + w\}\right) \geq m - 1.$$

Problem 2.3. Consider the set $S \subseteq \mathbb{C}^3$ of those vectors whose entries are either 0 or 1:

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{C}^3.$$

How many subsets of S form bases for $\mathbb{C}^3$?

3 For submission to Handong Park

Problem 3.1. Suppose that V is finite dimensional and that $U \leq V$ is a subspace with $\dim U = \dim V$. Show that $U = V$.

Problem 3.2. Suppose that V and W are finite dimensional vector spaces.

1. Show that there exists a surjective map $V \to W$ if and only if $\dim V \geq \dim W$.
2. Show that there exists an injective map $V \to W$ if and only if $\dim V \leq \dim W$.

Problem 3.3. For sets X, Y, and Z, there is an alternating sum formula

$$\begin{aligned} |X \cup Y \cup Z| = |X| + |Y| + |Z| \\ - |X \cap Y| - |X \cap Z| - |Y \cap Z| \\ + |X \cap Y \cap Z| \end{aligned}$$

expressing the cardinality of the union by accounting for the overlaps of the sets. By analogy, for subspaces X, Y, and Z of V you might also expect the formula

$$\begin{aligned} \dim(X + Y + Z) = \dim X + \dim Y + \dim Z \\ - \dim(X \cap Y) - \dim(X \cap Z) - \dim(Y \cap Z) \\ + \dim(X \cap Y \cap Z) \end{aligned}$$

to hold. Prove or disprove this formula.

4 For submission to Rohil Prasad

Problem 4.1. Let V_m denote the vector space of polynomials of degree at most m, and for each j suppose that f_j is some polynomial of degree j. Show that $\{f_0, f_1, \ldots, f_m\}$ form a basis for V_m.

Problem 4.2. Suppose that U and W are subspaces of $\mathbb{R}^8$ such that $\dim U = 3$, $\dim V = 5$, and $U + V = \mathbb{R}^8$. Show that $U + V$ is a *direct sum*.

Problem 4.3.
1. The complex numbers $\mathbb{C}$ can be considered as a vector space over the real numbers $\mathbb{R}$. What is its dimension as a real vector space?
2. Similarly, any complex vector space V can be considered as a real vector space V^- by only allowing multiplication by real scalars. If the complex vector space V is finite dimensional of dimension d, what dimension is the real vector space V^-?