Homework #3 Solutions

Thayer Anderson, Davis Lazowski, Handong Park, Rohil Prasad
Eric Peterson

October 3, 2016

1 For submission to Thayer Anderson

Problem 1.1. Let V be the vector space of polynomials of degree at most 3. Prove or disprove that there exists a basis of V consisting of polynomials, none of which are of degree 3.

Solution. Suppose $(f_1, \dots, f_4)$ is a collection of vectors in V consisting of polynomials f_j with $\deg f_j < 3$. In this case, each f_j admits an expression of the form

$$f_j = a_{j,0} + a_{j,1}x + a_{j,2}x^2,$$

and a linear combination of the f_js has the form

$$k_1 f_1 + \cdots + k_4 f_4 = (k_1 a_{1,0} + \cdots + k_4 a_{4,0}) + (k_1 a_{1,1} + \cdots + k_4 a_{4,1})x + (k_1 a_{1,2} + \cdots + k_4 a_{4,2})x^2.$$

Hence, the monomial x^3 is not in their span, so they cannot form a basis. (ECP)

Problem 1.2 (Follow-up to problem 3.2 from last time)**.** Exhibit an example of a vector space V with non-equal subspaces U_1, U_2, and U_3 such that

$$U_1 \oplus U_3 = U_2 \oplus U_3.$$

Solution. Let U_3 be the span of $(1, 0)$ in $\mathbb{R}^2$, and let U_1 and U_2 be any distinct runs of the complementation algorithm. (For example, U_1 can be the span of $(0, 1)$ and U_2 can be the span of $(1, 1)$.) We can re-express these subspaces as

$$U_3 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| y = 0 \right\}, \qquad U_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x = 0 \right\}, \qquad U_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x = y \right\}.$$

These sums are direct because a vector in $U_1 \cap U_3$ must have $x = 0$ and $y = 0$, and a vector in $U_2 \cap U_3$ must have $y = 0$ and $x = y$. Finally, these sums because we have $\dim(U \oplus V) = \dim U + \dim V$, and because we have solved Problem 3.1. (ECP)

Problem 1.3.

1. Under what conditions on the scalars $a, b \in \mathbb{C}$ are the vectors

$$\begin{pmatrix} 1 \\ a \end{pmatrix}, \begin{pmatrix} 1 \\ b \end{pmatrix}$$

a linearly dependent set in $\mathbb{C}^2$?

2. Under what conditions on the scalars $a, b, c \in \mathbb{C}$ are the vectors

$$\begin{pmatrix} 1 \\ a \\ a^2 \end{pmatrix}, \begin{pmatrix} 1 \\ b \\ b^2 \end{pmatrix}, \begin{pmatrix} 1 \\ c \\ c^2 \end{pmatrix}$$

a linearly dependent set in $\mathbb{C}^3$? (Come up with conditions analogous to those in the previous part.)

3. State and prove an analogous condition on sequences of n scalars forming sequences of n vectors in $\mathbb{C}^n$.

Solution. Parts 1 and 2 we will work by hand. There is a more "by-hand" solution to Part 3 given by Thayer below, but we will first give a "clever" solution that references some (simple) results further in the book.

1. If there is a linear dependence among two vectors, one must be a scalar multiple of the other. Since both of their first components are fixed at 1, the scalar must be 1, and hence it must be the case that $a = b$.

2. Again suppose that there is a linear dependence among the vectors:

$$k_1 \begin{pmatrix} 1 \\ a \\ a^2 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ b \\ b^2 \end{pmatrix} + k_3 \begin{pmatrix} 1 \\ c \\ c^2 \end{pmatrix} = 0.$$

   If any of the coefficients is zero, then we have reduced to the previous case, so suppose instead that all of the coefficients in the linear combination are nonzero. We will start to solve the linear system in k_1, k_2, k_3:

$$\begin{aligned} k_1 + k_2 + k_3 &= 0, \\ ak_1 + bk_2 + ck_3 &= 0, \\ a^2k_1 + b^2k_2 + c^2k_3 &= 0. \end{aligned}$$

   Immediately we see $k_3 = -(k_1 + k_2)$. The second equation then becomes

$$ak_1 + bk_2 + c(-k_1 - k_2) = 0,$$

   in which we solve for k_2 to get

$$k_2 = \frac{c-a}{b-c}k_1, \qquad k_3 = -(k_1 + k_2) = \frac{a-b}{b-c}k_1.$$

   (Or, if division by $b - c$ is not allowed, this is because $b = c$ and we are done.) Substituting everything into the final equation gives

$$\begin{aligned} a^2k_1 + b^2\frac{c-a}{b-c}k_1 + c^2\frac{a-b}{b-c}k_1 &= 0 \\ \frac{a^2b - a^2c + b^2c - b^2a + c^2a - c^2b}{b-c} &= 0, \end{aligned}$$

   which factors as

$$\frac{(a-b)(a-c)(b-c)}{b-c} = 0.$$

   The zero product property then forces either $a - b = 0$ or $a - c = 0$.

3. Take $(n + 1)$ points $x_0, \dots, x_n$ and consider the problem of constructing a degree n polynomial

$$f(x) = a_0 + a_1x + \cdots + a_nx^n$$

   with roots at these specified points. By setting $x = x_j$ for various j, we construct a linear system in the coefficients a_k:

$$\begin{aligned} 0 &= a_0 + a_1x_1 + \cdots + a_nx_1^n \\ 0 &= a_0 + a_1x_2 + \cdots + a_nx_2^n \\ &\vdots \\ 0 &= a_0 + a_1x_n + \cdots + a_nx_n^n, \end{aligned}$$

or, equivalently,

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^n & x_2^n & \cdots & x_n^n \end{pmatrix}.$$

We are looking for a linear dependence among the rows of this matrix. Axler 3.118 says that the rows of a matrix are linearly independent if and only if the columns are linearly independent, which connects with our homework problem.

Interpreting the problem as a linear system has advantages. We know that this system always has a solution: we can take $f(x) = \prod_{j=0}^{n}(x - x_j)$. Linear dependence of the rows then corresponds to when we can find multiple solutions to this problem. If some value is repeated in our list (say, $x_n = x_j$ for $j < n$), then $f(x) = \prod_{j=0}^{n-1}(x - x_j)$ also has roots in all the right places (since x_j already guaranteed $f(x_n) = f(x_j) = 0$), and we we can multiply by any one left-over factor $(x - s)$ that we like to get a polynomial of the right degree. Conversely, if all of the x_j are distinct, then the easy part of Axler's 4.17 (which is just the polynomial division algorithm) says that there is a unique solution to this polynomial interpolation problem. (ECP)

Solution. Here I will focus on an inductive solution for the last part of the problem. Suppose we have n vectors represented as follows:

$$\begin{pmatrix} 1 \\ a_1 \\ \vdots \\ a_1^{n-1} \end{pmatrix}, \begin{pmatrix} 1 \\ a_2 \\ \vdots \\ a_2^{n-1} \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ a_n \\ \vdots \\ a_n^{n-1} \end{pmatrix},$$

Suppose that for a set of $n-1$ vectors with length $n-1$ of the above form (except shorter by one entry) that linear dependence occurs if and only if $a_i = a_j$ for some $i \neq j$. We have verified the base case in parts 1 and 2. Suppose that the set of n vectors given above has some linear relation (not necessarily non-trival). Then we have:

$$c_1 \begin{pmatrix} 1 \\ a_1 \\ \vdots \\ a_1^{n-1} \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ a_2 \\ \vdots \\ a_2^{n-1} \end{pmatrix} + \cdots, +c_n \begin{pmatrix} 1 \\ a_n \\ \vdots \\ a_n^{n-1} \end{pmatrix} = 0$$

This generates the following system of n equations:

$$\begin{aligned} c_1 + c_2 + \cdots + c_n &= 0 \\ c_1a_1 + c_2a_2 + \cdots + c_na_n \\ &\vdots \\ c_1a_1^{n-1} + c_2a_2^{n-1} + \cdots + c_na_n^{n-1} &= 0 \end{aligned}$$

Taking the jth equation, multiplying by a_n and subtracting by the $j+1$ equation yields a new system of n equations.

$$\begin{aligned} &a_n(c_1 + c_2 + \cdots + c_n) \\ &\;-(c_1a_1 + c_2a_2 + \cdots + c_na_n) = 0 \\ &\vdots \\ &a_n(c_1a_1^{n-2} + \cdots + c_na_n^{n-2}) \\ &\;-(c_1a_1^{n-1} + c_2a_2^{n-1} + \cdots + c_na_n^{n-1}) = 0 \end{aligned}$$

The solutions to original equations must also be solutions to the new system of equations. Simplifying the new system of equations produces the following form:

$$\begin{aligned} c_1(a_n - a_1) + \cdots + c_{n-1}(a_n - a_{n-1}) + 0 &= 0 \\ &\vdots \\ c_1(a_n a_1^{n-1} - a_1^n) + \cdots + c_{n-1}(a_n a_{n-1}^{n-1} - a_{n-1}^n) + 0 &= 0 \end{aligned}$$

And factoring out the largest powers:

$$\begin{aligned} c_1(a_n - a_1) + \cdots + c_{n-1}(a_n - a_{n-1}) + 0 &= 0 \\ &\vdots \\ c_{n-1}(a_n - a_1)a_1^{n-1} + \cdots + c_{n-1}(a_n - a_{n-1})a_{n-1}^{n-1} + 0 &= 0 \end{aligned}$$

We recognize that this system of equations can be represented as a linear relation on the following $n-1$ vectors:

$$c_1(a_n - a_1)\begin{pmatrix} 1 \\ a_1 \\ \vdots \\ a_1^{n-1} \end{pmatrix} + c_2(a_n - a_2)\begin{pmatrix} 1 \\ a_2 \\ \vdots \\ a_2^{n-1} \end{pmatrix} + c_{n-1}(a_n - a_{n-1})\begin{pmatrix} 1 \\ a_{n-1} \\ \vdots \\ a_{n-1}^{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Such a linear combination exactly satisfies the inductive hypothesis. We know that the linear combination is non-trivial exactly when $a_i = a_j$ for some $1 \leq i, j \leq n-1$ with $i \neq j$. In that case, our condition is proven. If there is linear independence amongst these $n-1$ vectors, then each scalar is equal to 0. Supposing that the original n vectors are linearly dependent, at least one $c_i \neq 0$. Meaning that $a_n = a_i$ for some $i < n$. Hence, our condition is proven. This means that if the original n vectors are linearly dependent, it is necessary that $a_i = a_j$ for some $i \neq j$. If $a_i = a_j$ for $i \neq j$ then two of the vectors in the original set are identical, and therefore the set is linearly dependent. This proves sufficiency of our condition and thereby completes the proof of the claim. Credit to Luke for first showing me this elegant method. (Thayer)

2 For submission to Davis Lazowski

Problem 2.1. For two subspaces U and W of a vector space V, show that if every vector in V belongs to either U or W (or both) then it must be the case that $U = V$ or $W = V$ (or both).

Solution. If $V = U = W$, then done.

Otherwise, without loss of generality let $U \neq V$, so that there exists $w \in V$, $w \notin U$. Then $w \in W$.

Suppose there existed $u \in U, u \notin W$. Then consider $u + w$. If $u + w \in U$, then by additive closure $(u + w) - u \in U \implies w \in U$, so $(u + w) \notin U$. If $u + w \in W$, then $(u + w) - w \in W \implies u \in W$, so $(u + w) \notin W$. But this is a contradiction. Therefore there does not exist $u \in U, u \notin W$. Therefore $U \subset W$, and because every $v \in V$ is in U or W, every $v \in V$ must be in W, so $W = V$. (DL)

Problem 2.2. Suppose that $v_1, \ldots, v_m \in V$ form a linearly independent set and let $w \in V$ be another vector. Show that

$$\dim\left(\operatorname{span}\{v_1 + w, \ldots, v_m + w\}\right) \geq m - 1.$$

Solution. Suppose $\dim\operatorname{span}\{\mathrm{v_1 + w ... v_m + w}\} \leq \mathrm{m-2}$. Then there exists c_{ij}, and $z_1 ... z_{m-2}$, such that

$$\sum_{j=1}^{m-2} c_{ij} z_j = v_i + w$$

so that

$$\sum_{j=1} (c_{ij} - c_{i'j}) z_j = (v_i + w) - (v_i' + w) = v_i - v_i'.$$

So, $\{v_1 - v_m, v_2 - v_m ... v_{n-1} - v_m\} \subset \{z_1 ... z_{m-2}\}$, so, in particular, $\dim\operatorname{span}\{\mathrm{v_1 - v_m, v_2 - v_m ... v_{n-1} - v_m}\} \leq \mathrm{m-2}$, which implies that $\{v_1 - v_m .... v_{m-1} - v_m\}$ is not a linearly independent list. Therefore, with not all $\alpha_j = 0$,

$$\sum_{j=1}^{m-1} \alpha_j (v_j - v_n) = 0.$$

Defining $\beta_j = \alpha_j$, and $\sum_{j=1}^{m-1} -\alpha_j = \beta_m$, where not all β_j are zero because they are simply the same as the α_j,

$$\sum_{j=1}^{m} \beta_j v_j = 0$$

contradicting the linear independence of the v_j. (DL)

Problem 2.3. Consider the set $S \subseteq \mathbb{C}^3$ of those vectors whose entries are either 0 or 1:

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{C}^3.$$

How many subsets of S form bases for $\mathbb{C}^3$?

Solution. The 0 vector is not linearly independent with itself and so can be discounted.

There are three cases:

1. **Choosing two or more of the standard basis vectors**: Then we can choose any third vector except their sum, so this is $\binom{3}{2} * 4 - 2 = 10$ choices (the -2 coming because otherwise we get three copies of the standard basis vectors).

2. **Choosing two or more of the combinations of two basis vectors**: We can choose any of the other vectors. So we get $\binom{3}{2} * 5 - 2 = 13$ choices.

3. **Choosing one with 1 one, one with 2 ones, one with 3 ones**

   After choosing the $\hat{x} + \hat{y} + \hat{z}$ and one with one standard basis vector, say $\hat{e}_1$, the only vector in our span is $\hat{e}_2 + \hat{e}_3$. So there are two more choices. So there are $3 * 2 = 6$ choices. So there are 29 possible choices. (DL)

3 For submission to Handong Park

Problem 3.1. Suppose that V is finite dimensional and that $U \leq V$ is a subspace with $\dim U = \dim V$. Show that $U = V$.

Solution. U is a subspace of V which is finite dimensional, so U is finite dimensional. We have that $\dim V = n$ for some $n \in \mathbb{N}$, and moreover, since $\dim U = \dim V$, we also know that $\dim U = n$. So take a basis for U, consisting of $u_1, u_2, ..., u_n \in U$ which are linearly independent. However, since $U \leq V$, we have that for all i such that $1 \leq i \leq n$, $u_i \in V$. Thus these u_i also form a basis for V, since we have n linearly independent vectors in V, which has dimension n. We then have

$$U = \text{Span}(\{u_1, ..., u_n\}) = V \tag{HP}$$

Problem 3.2. Suppose that V and W are finite dimensional vector spaces.

1. Show that there exists a surjective map $V \to W$ if and only if $\dim V \geq \dim W$.
2. Show that there exists an injective map $V \to W$ if and only if $\dim V \leq \dim W$.

Solution. 1. First, we prove that if there exists such a surjective map, then $\dim V \geq \dim W$.
We have $f : V \to W$ surjective, thus by the Rank-Nullity Theorem:

$$\dim V = \dim Im(f) + \dim N(f)$$

where $Im(f)$ is the image of V under f and $N(f)$ is the null space. Since dimension for any subspace is at least 0, we have that $\dim N(f) \geq 0$. However, since f is surjective, we must have that $Im(f) = W$, so we have

$$\dim V = \dim W + \dim N(f) \geq \dim W$$

Now, we prove the other direction. Suppose $\dim V \geq \dim W$. Call $\dim V = n$ and $\dim W = m$, then $n \geq m$.
We will construct a surjective map $g : V \to W$. Take a basis $v_1, ..., v_n$ for V, and a basis $w_1, ..., w_m$ for W. Set $g(v_i) = w_i$ for all i such that $1 \leq i \leq m$. For any remaining extra basis vectors $v_m, ..., v_n$, just set $g(v_i) = 0$ for all i such that $n \geq i > m$.
Now, given $w \in W$, we want to show that there exists $v \in V$ such that $g(v) = w$. Since we have a basis for W, we can express w in terms of the basis vectors:

$$w = a_1 w_1 + a_2 w_2 + ... + a_m w_m$$

where the $a_i \in K$ are scalars. Then we have

$$g^{-1}(w) = a_1 v_1 + a_2 v_2 + ... + a_m w_m \in V$$

This works - by linearity,

$$g(a_1 v_1 + a_2 v_2 + ... + a_m w_m) = a_1 g(v_1) + ... + a_m g(w_m) = a_1 w_1 + ... + a_m w_m = w$$

Thus we've shown that for any $w \in W$, we can find $v \in V$ such that $g(v) = w$, so g is surjective, and thus a $g : V \to W$ surjective map exists.

2. First, we prove that if there exists an injective map, then $\dim V \leq \dim W$.
Since $f : V \to W$ is injective, then $N(f)$, the null space of f, must just be 0. Thus $\dim N(f) = 0$, and by Rank-Nullity:

$$\dim V = \dim Im(f) + \dim N(f) = \dim Im(f)$$

Since the image $Im(f)$ is a subspace of W, we have that $\dim Im(f) \leq \dim W$, so

$$\dim V = \dim Im(f) \leq \dim W$$

Now, we prove the other direction. Suppose $\dim V \leq \dim W$. Call $\dim V = n$ and $\dim W = m$, then $n \leq m$.
We will construct an injective map $g : V \to W$. Take a basis $v_1, ..., v_n$ for V, and a basis $w_1, ..., w_m$ for

W. Set $g(v_i) = w_i$ for all i such that $1 \leq i \leq n$.
This map is injective. To prove this, suppose $g(a) = 0$. We want to prove that $a = 0$.
Write a as a linear combination of basis vectors, then $a = c_1 v_1 + ... + c_n v_n$. But then,

$$g(a) = g(c_1 v_1 + ... + c_n v_n) = c_1 g(v_1) + ... + c_n g(v_n) = c_1 w_1 + ... + c_n w_n$$

Since $w_1, ..., w_n$ are linearly independent in W, we must have that $c_1, ..., c_n = 0$. Thus $a = 0$, so the only way to have $g(a) = 0$ is by having $a = 0$, meaning that the null space of g is also 0, making g injective. (HP)

Problem 3.3. For sets X, Y, and Z, there is an alternating sum formula

$$\begin{aligned} |X \cup Y \cup Z| = |X| &+ |Y| + |Z| \\ &- |X \cap Y| - |X \cap Z| - |Y \cap Z| \\ &+ |X \cap Y \cap Z| \end{aligned}$$

expressing the cardinality of the union by accounting for the overlaps of the sets. By analogy, for subspaces X, Y, and Z of V you might also expect the formula

$$\begin{aligned} \dim(X + Y + Z) = \dim X &+ \dim Y + \dim Z \\ &- \dim(X \cap Y) - \dim(X \cap Z) - \dim(Y \cap Z) \\ &+ \dim(X \cap Y \cap Z) \end{aligned}$$

to hold. Prove or disprove this formula.

Solution. This formula fails. Consider the overall vector space $\mathbb{R}^2$, with

$$\begin{aligned} X &= \{(x, 0) \mid x \in \mathbb{R}\} \\ Y &= \{(0, y) \mid y \in \mathbb{R}\} \\ Z &= \{(x, y) \mid x = y \in \mathbb{R}\} \end{aligned}$$

In this situation, we find that

$$\begin{aligned} X \cap Y &= \{0\} \\ X \cap Z &= \{0\} \\ Y \cap Z &= \{0\} \\ X \cap Y \cap Z &= \{0\} \end{aligned}$$

Then we get that A basis for X is $\{(1, 0)\}$, a basis for Y is $\{(0, 1)\}$, and a basis for Z is $\{(1, 1)\}$.
So according to the formula, we should have

$$\begin{aligned} \dim(X + Y + Z) &= \dim X + \dim Y + \dim Z \\ &\quad - \dim(X \cap Y) - \dim(X \cap Z) - \dim(Y \cap Z) \\ &\quad + \dim(X \cap Y \cap Z) \\ &= 1 + 1 + 1 - 0 - 0 - 0 + 0 \\ &= 3 \end{aligned}$$

But we know that $X + Y + Z = \mathbb{R}^2$, so $\dim(X + Y + Z) = 2$, disproving the claim. (HP)

4 For submission to Rohil Prasad

Problem 4.1. Let V_m denote the vector space of polynomials of degree at most m and for each j suppose that f_j is some polynomial of degree j. Show that $\{f_0, f_1, \ldots, f_m\}$ form a basis for V_m.

Solution. We will show this by induction on m.

For $m = 0$, the assertion is clear. We have $V_0 = \mathbb{R}$ and any constant $f_0 \in \mathbb{R}$ is a basis.

Now assume that any set of polynomials $g_0, g_1, \ldots, g_{m-1}$ form a basis of V_{m-1} where g_i has degree i.

To show that $f_0, f_1, \ldots, f_m$ form a basis for V_m, we must show that they are linearly independent and span V_m.

Let $p = \sum_{i=0}^{m} c_i x^i$ be a polynomial and element of V_m. Since f_m is of degree m, it has a nonzero x^m-coefficient which we will denote by c.

It follows that the polynomial $p - \frac{c_m}{c} f \in V_m$ has degree $\leq m-1$ since its x^m-coefficient is 0. By our inductive hypothesis, $f_0, \ldots, f_{m-1}$ span the space of polynomials of degree at most $m-1$. Therefore, $p - \frac{c_m}{c} f$ is equal to a linear combination $\sum_{i=0}^{m-1} b_i f_i$. As a result, $p = \frac{c_m}{c} f + \sum_{i=0}^{m-1} b_i f_i$, so p can be expressed as a linear combination of the f_i and they span V_m.

Now let λ_i be scalars such that $\sum_{i=0}^{m} \lambda_i f_i = 0$. Showing linear independence is equivalent to showing that all the λ_i must be 0. Since f_m is the only polynomial of degree m in the set, the sum $\sum_{i=0}^{m} \lambda_i f_i$ has the same x^m-coefficient as $\lambda_m f_m$. However, since the right-hand side is equal to 0, this x^m-coefficient is 0 and therefore λ_m must equal 0.

Plugging this into the original sum, we find that $\sum_{i=0}^{m-} \lambda_i f_i = 0$. However, by our inductive hypothesis, $f_0, \ldots, f_{m-1}$ are linearly dependent and therefore the rest of the λ_i must be 0 as well. (RP)

Problem 4.2. Suppose that U and V are subspaces of $\mathbb{R}^8$ such that $\dim U = 3$, $\dim V = 5$, and $U + V = \mathbb{R}^8$. Show that $U + V$ is a *direct sum.*

Solution. By definition, it suffices to show that $U \cap V = \{0\}$.

Let u_1, u_2, u_3 be a basis of U and let v_1, v_2, v_3, v_4, v_5 be a basis of V.

Since $U + V = \mathbb{R}^8$, we have that any vector in $\mathbb{R}^8$ can be expressed as a sum of a vector in U and a vector in V. Therefore, any vector in $\mathbb{R}^8$ can be expressed as a linear combination of the set $\{u_1, u_2, u_3, v_1, v_2, v_3, v_4, v_5\}$. In other words, this set spans $\mathbb{R}^8$. Since this set spans $\mathbb{R}^8$ and has size equal to the dimension of $\mathbb{R}^8$, it follows that it is a basis of $\mathbb{R}^8$, and therefore linearly independent.

Any vector $w \in U \cap V$ can be expressed as a linear combination $\sum_{i=1}^{3} a_i u_i$ and $\sum_{j=1}^{5} b_j v_j$ since the u_i and v_j are bases of theri respective spaces.

It follows that $\sum_{i=1}^{3} a_i u_i - \sum_{j=1}^{5} b_j v_j = 0$. By the linear independence of these 8 vectors, all of the scalar coefficients are 0 and therefore $w = 0$. (RP)

Problem 4.3.

1. The complex numbers $\mathbb{C}$ can be considered as a vector space over the real numbers $\mathbb{R}$. What is its dimension as a real vector space?

2. Similarly, any complex vector space V can be considered as a real vector space V^- by only allowing multiplication by real scalars. If the complex vector space V is finite dimensional of dimension d, what dimension is the real vector space V^-?

Solution. 1. $\mathbb{C}$ has dimension 2 as a vector space. We will show this by showing $\{1, i\}$ is a basis.

Any complex number can by definition be written as $a + bi$ for $a, b \in \mathbb{R}$, so this set spans $\mathbb{C}$.

Any linear combination $a + bi$ is equal to 0 iff the real and imaginary parts are equal to 0, which implies $a = b = 0$ and therefore they are linearly independent.

2. Let $S = \{v_1, \ldots, v_n\{$ be a basis of V.

We will show that the vectors $S^- = \{v_1, \ldots, v_n, i \cdot v_1, \ldots, i \cdot v_n\}$ are a basis of V^-, and as a result show that it has dimension $2d$.

Any vector $v \in V$ can be expressed as a sum

$$v = \sum_{k=1}^{d} (a_k + b_k i) v_k$$

for $a_k, b_k \in \mathbb{R}$ by definition. However, V and V^- are the same as sets.

It follows by the distributivitiy of multiplication by scalars that

$$v = \sum_{k=1}^{d} a_k v_k + b_k (i \cdot v_k)$$

so V^- is spanned by S^-.

Now assume $\sum_{k=1}^{d} a_k v_k + b_k (i \cdot v_k) = 0$ for some scalars $a_k, b_k \in \mathbb{R}$.

We can use distributivity of multiplication by scalars to find that this is equivalent to $\sum_{k=1}^{d} (a_k + b_k i) v_k = 0$ in V. However, since S is linearly independent, we have $a_k + b_k i = 0$ for every k. This in turn implies $a_k, b_k = 0$ for every k and so S^- is linearly independent as well. (RP)