Homework #2

Math 25a

Due: September 21, 2016

Guidelines:

- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper (stapling together multiple pages if necessary).
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.
- For the first few assignments, we would like you to keep track of how long it takes you to complete the (whole) assignment and to notate that somewhere on your submission, so that we can calibrate the difficulty of the homeworks.

1 For submission to Thayer Anderson

Problem 1.1. Let E denote the *extended reals*:

$$E := \mathbb{R} \cup \{-\infty, \infty\}.$$

The usual arithmetic operations on $\mathbb{R}$ can be extended to E by

$$t \cdot \infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t \cdot (-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\infty + t = \begin{cases} \infty & \text{if } t \neq -\infty, \\ 0 & \text{if } t = -\infty, \end{cases} \qquad (-\infty) + t = \begin{cases} -\infty & \text{if } t \neq \infty, \\ 0 & \text{if } t = \infty. \end{cases}$$

Show that E *fails* to be a field.

Problem 1.2.
1. Give an example of a subset $U \subseteq \mathbb{R}^2$ such that U is closed under addition and taking additive inverses, yet U is not a subspace.
2. Give an example of a subset $V \subseteq \mathbb{R}^2$ such that V is closed under scalar multiplication, and yet V is not a subspace.

Problem 1.3. Show that $U = \{(x, x, y, y) \in K^4 \mid x, y \in K\}$ forms a subspace of K^4. Then, exhibit a second subspace W so that there is a direct sum decomposition

$$K^4 = U \oplus W.$$

2 For submission to Davis Lazowski

Problem 2.1. Complex numbers $z \in \mathbb{C}$ can be written as $z = a + bi$ for $a, b \in \mathbb{R}$ real numbers. Show that multiplicative inverses exist in $\mathbb{C}$, i.e., for every nonzero $z = a+bi$ there is an element $z^{-1} = c+di$ satisfying $z \cdot z^{-1} = 1$. Give formulas for c and d in terms of a and b.

Problem 2.2. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *even* if it satisfies $f(-x) = f(x)$ and *odd* if it satisfies $f(-x) = -f(x)$.

1. Show that the collections of even functions U_{even} and odd functions U_{odd} both form subspaces of the vector space $\mathbb{R}^{\mathbb{R}}$ of all functions.
2. Show that there is a direct sum decomposition
$$\mathbb{R}^{\mathbb{R}} \cong U_{\text{even}} \oplus U_{\text{odd}}.$$

Problem 2.3. 1. Is $\{(a, b, c) \in \mathbb{R}^3 \mid a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?

2. Is $\{(a, b, c) \in \mathbb{C}^3 \mid a^3 = b^3\}$ a subspace of $\mathbb{C}^3$?

3 For submission to Handong Park

Problem 3.1. For each of the following subsets of K^3, check whether they form a subspace:

1. $\{(x_1, x_2, x_3) \in K^3 \mid x_1 + 2x_2 + 3x_3 = 0\}$.
2. $\{(x_1, x_2, x_3) \in K^3 \mid x_1 + 2x_2 + 3x_3 = 4\}$.
3. $\{(x_1, x_2, x_3) \in K^3 \mid x_1 x_2 x_3 = 0\}$.
4. $\{(x_1, x_2, x_3) \in K^3 \mid x_1 = 5x_3\}$.

Problem 3.2. 1. Does the operation of addition of subspaces have an identity? (That is: for a vector space V, is there a fixed subspace $W \subseteq V$ such that $U + W = U$ for any subspace $U \subseteq V$?)

2. Which subspaces admit additive inverses? Give an example of a vector space V with subspaces U_1, U_2, and W such that $U_1 + W = U_2 + W$ and yet $U_1 \neq U_2$.

Problem 3.3. Consider the example from class of *clock arithmetic* or *modular arithmetic* with modulus n: two integers a and b are called "equivalent" if a differs from b by a multiple of n. In this case, we write
$$a \equiv b \pmod{n}.$$

1. Show that n is composite (i.e., not prime) if and only if there are clock positions
$$a \not\equiv 0, \quad b \not\equiv 0 \pmod{n}$$
such that $a \cdot b \equiv 0 \pmod{n}$.
2. Now let n be prime and consider an integer a with $a \not\equiv 0 \pmod{n}$.
   (a) Show that the function
   $$f\colon \left\{ \begin{array}{c} \text{clock faces} \\ \text{with } n \text{ positions} \end{array} \right\} \to \left\{ \begin{array}{c} \text{clock faces} \\ \text{with } n \text{ positions} \end{array} \right\}$$
   defined by $f(b) \equiv a \cdot b \pmod{n}$ is *injective*.
   (b) Use the finiteness of clock faces to show that f is also *surjective*.
   (c) Conclude that there is a clock face b with $a \cdot b \equiv 1 \pmod{n}$.
3. Suppose that $n = 4 = 2 \cdot 2$, an example of a composite value of n. Set $a = 2$ and compute all the values of $f(b)$.

4 For submission to Rohil Prasad

Problem 4.1. Explain why there does *not* exist $\lambda \in \mathbb{C}$ such that

$$\lambda \cdot \begin{pmatrix} 2-3i \\ 5+4i \\ -6+7i \end{pmatrix} = \begin{pmatrix} 12-5i \\ 7+22i \\ -32-9i \end{pmatrix}.$$

Problem 4.2. Suppose that U_1 and U_2 are subspaces of a vector space V. Show that $U_1 \cup U_2$ is also a subspace if and only if one of the two subspaces is contained in the other.

Problem 4.3. In the definition of a vector space [Axler 1.19], there is the following condition:

> For each vector v there is an additive inverse vector $-v$ satisfying $v + (-v) = 0$.

Show that this condition can be replaced by the condition

$$0 \cdot v = 0.$$

(That is, show that either equation follows from the other using the other axioms of a vector space.)

Problem 4.4. A function $f \colon \mathbb{R} \to \mathbb{R}$ is said to be *periodic (with period $p \neq 0$)* if it satisfies

$$f(x+p) = f(x).$$

1. Fix a $p \neq 0$. Does the set of functions which are periodic with period p form a subspace of the vector space of all functions?
2. Now consider the set of *all* periodic functions, with unspecified period. (This is the union over all possible values of p of the sets considered in the first part.) Does this form a subspace of the vector space of all functions?