Homework #2 Solutions

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1 For submission to Thayer Anderson

Problem 1.1. Let E denote the *extended reals*:

$$E := \mathbb{R} \cup \{-\infty, \infty\}.$$

The usual arithmetic operations on $\mathbb{R}$ can be extended to E by

$$t \cdot \infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t \cdot (-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\infty + t = \begin{cases} \infty & \text{if } t \neq -\infty, \\ 0 & \text{if } t = -\infty, \end{cases} \qquad (-\infty) + t = \begin{cases} -\infty & \text{if } t \neq \infty, \\ 0 & \text{if } t = \infty. \end{cases}$$

Show that E *fails* to be a field.

Solution. Using the distributive property, we have the following series of equalities:

$$\begin{aligned} \infty(1-2) &= \infty - 2 \cdot \infty \\ &= \infty - \infty = 0 \end{aligned}$$

And instead of distributing, we may do the subtraction first:

$$\infty(1-2) = \infty(-1) = -\infty$$

Thus we have $-\infty = 0$ which is a contradiction as $-\infty + 1 \neq 1$. (This shows that $-\infty$ is not the additive identity). (TA)

Problem 1.2. 1. Give an example of a subset $U \subseteq \mathbb{R}^2$ such that U is closed under addition and taking additive inverses, yet U is not a subspace.

2. Give an example of a subset $V \subseteq \mathbb{R}^2$ such that V is closed under scalar multiplication, and yet V is not a subspace.

Solution. 1. Consider $U = \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{Z}\}$. Suppose $(n, 0) \in U$ and $(m, 0) \in U$ arbitrary. Then $(n, 0) + (m, 0) = (n + m, 0)$ and $n + m$ is also an integer as n and m are integers. Additionally, we have additive inverses. If $n \in \mathbb{Z}$ then $-n \in \mathbb{Z}$ and $(n, 0) + (-n, 0) = (0, 0)$.

U is not closed under scalar multiplication. Take $(1, 0) \in U$ and $1/2 \in \mathbb{R}$. We see that $1/2(1, 0) = (1/2, 0) \notin U$.

2. Let $U = \{(x, y) \in \mathbb{R}^2 | x = 0 \text{ or } y = 0\}$. Then suppose $(x, y) \in U$ for some fixed $x, y \in \mathbb{R}$. Then $c(x, y) = (cx, cy)$. Without loss of generality, say that $y = 0$ then $(cx, cy) = (cx, 0) \in U$.

   U is not a subspace because $(1, 0) + (0, 1) = (1, 1) \notin U$. (TA)

Problem 1.3. Show that $U = \{(x, x, y, y) \in K^4 \mid x, y \in K\}$ forms a subspace of K^4. Then, exhibit a second subspace W so that there is a direct sum decomposition

$$K^4 = U \oplus W.$$

Solution. Suppose that $v = (x, x, y, y) \in U$ and $w = (x', x', y', y') \in U$. Then $v + w = (x + x', x + x', y + y', y + y')$. We recognize that $v + w \in U$ and therefore U is closed under addition. Suppose that $c \in K$. Then $c \cdot v = (cx, cx, cy, cy)$. Again, cv satisfies the criteria of existence in U and therefore U is closed underscalar multiplication. We also verify that $0 = 0$ so $(0, 0, 0, 0) \in U$ and therefore U is non-empty. Thus U is a subspace of K^4.

Let $W = \{(a, 0, b, 0) | a, b \in K\}$. First we verify that W is a subspace. We see that $0 \in K$ so $(0, 0, 0, 0) \in W$ and W is non-empty. Let a, a', b, b', c be in K arbitrary. We note that $(a, 0, b, 0) + c(a', 0, b', 0) = (a + ca', 0, b + cb', 0)$. Since $a + ca', b + cb' \in K$ it follows that W is a subspace. (Note that I combined the scalar multiplication and addition criteria. If you are unsure this works, prove it!)

Next, we consider $U \cap W$. Suppose $v = (a, b, c, d) \in U \cap W$. Then $b = d = 0$ because $v \in W$. Furthermore $a = b$ and $c = d$ because $v \in U$. Therefore $v = 0$. Thus it makes sense to use the direct sum symbol to consider $U \oplus W$.

To prove that $U \oplus W = K^4$ then suppose $v = (a, b, c, d) \in K^4$ arbitrary. We want to show that $v = u + w$ for some $u \in U$ and some $w \in W$. Let

$$u = (b, b, d, d)$$
$$v = (a - b, 0, c - d, 0)$$

We observe that $u \in U$ and $v \in V$. Now we verify:

$$u + v = (a + b - b, b, c + d - d, d) = (a, b, c, d))$$

Thus $V = U \oplus W$ and the proof is complete. (TA)

2 For submission to Davis Lazowski

Problem 2.1. Complex numbers $z \in \mathbb{C}$ can be written as $z = a + bi$ for $a, b \in \mathbb{R}$ real numbers. Show that multiplicative inverses exist in $\mathbb{C}$, i.e., for every nonzero $z = a + bi$ there is an element $z^{-1} = c + di$ satisfying $z \cdot z^{-1} = 1$. Give formulas for c and d in terms of a and b.

Solution. Observe that $(a + bi)(a - bi) = a^2 + b^2$. Therefore

$$(a + bi)\frac{a - bi}{a^2 + b^2} = 1$$

Therefore

$$c := \frac{a}{a^2 + b^2}$$
$$d := -\frac{b}{a^2 + b^2}$$

(DL)

Problem 2.2. A function $f \colon \mathbb{R} \to \mathbb{R}$ is called *even* if it satisfies $f(-x) = f(x)$ and *odd* if it satisfies $f(-x) = -f(x)$.

1. Show that the collections of even functions U_{even} and odd functions U_{odd} both form subspaces of the vector space $\mathbb{R}^{\mathbb{R}}$ of all functions.

2. Show that there is a direct sum decomposition

$$\mathbb{R}^{\mathbb{R}} \cong U_{\text{even}} \oplus U_{\text{odd}}.$$

Solution. **Part a**

For even functions

It's enough to show that $f, g \in U_{\text{even}} \implies f + \lambda g \in U_{\text{even}}$.

But

$$(f + \lambda g)(-x) = f(-x) + \lambda g(-x) = f(x) + \lambda g(x) = (f + \lambda g)(x)$$

For odd functions

Likewise,

$$(f + \lambda g)(-x) = f(-x) + \lambda g(-x) = -f(x) - \lambda g(x) = -(f + \lambda g)(x)$$

Part b

First, I show that, for $f \in \mathbb{R}^{\mathbb{R}}$, then there exists $f_o \in U_{\text{odd}}, f_e \in U_{\text{even}}$, such that $f = f_o + f_e$.

Define

$$f_o(x) := \frac{f(x) - f(-x)}{2}$$
$$f_e(x) := \frac{f(x) + f(-x)}{2}$$

Then $f_o + f_e = f$, and

$$f_o(-x) = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -f_o(x)$$
$$f_e(-x) = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = f_e(x)$$

So f_o is odd and f_e is even, as required.

Next, suppose $f = f_o + f_e = f'_o + f'_e$. Then

$$\frac{f(x) + f(-x)}{2} = f_e = f'_e$$

and

$$\frac{f(x) - f(-x)}{2} = f_o = f'_o$$

So f_o, f_e are unique. (DL)

Problem 2.3. 1. Is $\{(a, b, c) \in \mathbb{R}^3 \mid a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?

2. Is $\{(a, b, c) \in \mathbb{C}^3 \mid a^3 = b^3\}$ a subspace of $\mathbb{C}^3$?

Solution. **Part a**

In $\mathbb{R}$, this condition is equivalent to $a = b$, because a^3 preserves sign, and $p : \mathbb{R}^+ \to \mathbb{R}^+$, $p : x \to x^{\frac{1}{3}}$ is injective.

Therefore

$$\begin{aligned}&(a, a, c) + \lambda(b, b, \tilde{c})\\ &= (a, a, c) + (\lambda b, \lambda b, \lambda\tilde{c})\\ &= (a + \lambda b, a + \lambda b, c + \lambda\tilde{c}) \in \{(a, b, c) \in \mathbb{R}^3 | a^3 = b^3\}\end{aligned}$$

Part b
No. $-1 = e^{i\pi}$, so that $(e^{i\frac{\pi}{3}})^3 = -1$.
Then $(e^{i\frac{\pi}{3}}, e^{i\frac{\pi}{3}}, 0), (1, 1, 0) \in \{(a, b, c) \in \mathbb{R}^3 | a^3 = b^3\}$. Yet

$$\begin{aligned}(e^{i\frac{\pi}{3}}, -1, 0) + (1, 1, 0) \\ = (e^{i\frac{\pi}{3}} + 1, 0, 0)\end{aligned}$$

But

$$(e^{i\frac{\pi}{3}} + 1)^3 = (e^{i\frac{2\pi}{3}} + 2e^{i\frac{\pi}{3}} + 1)(e^{i\frac{\pi}{3}} + 1) = (-1 + 3e^{i\frac{2\pi}{3}} + 3e^{i\frac{\pi}{3}} + 1) = 3e^{i\frac{\pi}{3}}(1 + e^{i\frac{\pi}{3}}) \neq 0 \qquad \text{(DL)}$$

3 For submission to Handong Park

Problem 3.1. For each of the following subsets of K^3, check whether they form a subspace:

1. $\{(x_1, x_2, x_3) \in K^3 \mid x_1 + 2x_2 + 3x_3 = 0\}$.
2. $\{(x_1, x_2, x_3) \in K^3 \mid x_1 + 2x_2 + 3x_3 = 4\}$.
3. $\{(x_1, x_2, x_3) \in K^3 \mid x_1x_2x_3 = 0\}$.
4. $\{(x_1, x_2, x_3) \in K^3 \mid x_1 = 5x_3\}$.

Solution. 1. Yes, this is a subspace. First, we show that the zero vector $(0, 0, 0)$ is in the set - which must be true since if we let $x_1 = 0, x_2 = 0, x_3 = 0$, we have

$$x_1 + 2x_2 + 3x_3 = 0 + 2(0) + 3(0) = 0$$

so that $(0, 0, 0)$ is in our subset of K^3.
Next, we check that the sum of any two vectors in our subset is also contained in the subset. Take (x_1, x_2, x_3) and (y_1, y_2, y_3) from our subset, then

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

We then have that

$$\begin{aligned}(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) &= x_1 + y_1 + 2x_2 + 2y_2 + 3x_3 + 3y_3 \\ &= (x_1 + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3) \\ &= 0 + 0 = 0\end{aligned}$$

showing that the sum of the two vectors also satisfies the condition for being in the subset.
Finally, we check that the scalar multiplication of any vector in our subset is also contained in the subset. For $c \in K$, we have

$$c(x_1, x_2, x_3) = (cx_1, cx_2, cx_3)$$

But then

$$cx_1 + 2cx_2 + 3cx_3 = c(x_1 + 2x_2 + 3x_3) = 0$$

showing that any scalar multiples are also contained. By proving these three properties, we now know that we indeed have not only a subset but a subspace of K^3.

2. No, this is NOT a subspace. The zero vector $(0, 0, 0)$ is not in the set, since

$$0 + 2(0) + 3(0) = 0 \neq 4$$

3. No, this is NOT a subspace. Take $(1,0,1)$ and $(0,1,0)$, which both satisfy $x_1x_2x_3 = 0$. However, $(1,0,1)+(0,1,0) = (1,1,1)$ which does not satisfy $x_1x_2x_3 = 0$, so the set is not closed under addition.

4. Yes, this is a subspace. First, we have that the zero vector is present, which is clear since if $x_1 = 0, x_2 = 0, x_3 = 0$, then $x_1 = 0 = 5(0) = 5x_3$.
   Take the sum of any vectors (x_1, x_2, x_3) and (y_1, y_2, y_3) that are in the subset, then we have the sum $(x_1+y_1, x_2+y_2, x_3+y_3)$. Since $x_1 = 5x_3$ and $y_1 = 5y_3$, $x_1+y_1 = 5x_3+5y_3 = 5(x_3+y_3)$, which shows closure underaddition.
   Now take any scalar $c \in K$, then we have $c(x_1, x_2, x_3) = (cx_1, cx_2, cx_3)$. Any such scalar multiple is also in the subset since $x_1 = 5x_3$ implies that $cx_1 = 5cx_3$, showing closure under scalar multiplication.
   By these three properties, we then have that this subset is a subspace of K^3. (HP)

Problem 3.2. 1. Does the operation of addition of subspaces have an identity? (That is: for a vector space V, is there a fixed subspace $W \subseteq V$ such that $U + W = U$ for any subspace $U \subseteq V$?)

2. Which subspaces admit additive inverses? Give an example of a vector space V with subspaces U_1, U_2, and W such that $U_1 + W = U_2 + W$ and yet $U_1 \neq U_2$.

Solution. 1. To find an identity for the operation of addition of subspaces, we must ask ourselves: given any vector space V, is there some subspace $W \subset V$ such that no matter what V we chose, $W \subset U$ and $U + W = U$ for any $U \subset V$?
   The answer is yes. The identity in question is precisely $\{0\}$, which must be part of any subspace $U \subset V$. $U + \{0\} = U$ for all $U \subset V$, so $\{0\}$ is our additive identity.

2. There is only one subspace that has an additive inverse - and that is $\{0\}$.
   Why? Consider any non-zero subspace $U \subset V$, and any other subspace $W \subset V$. We want to find W such that $U + W = \{0\}$.
   But if U is non-zero, there exists some $u \in U$ such that $u \neq 0$. Then it must be the case that $U + W$ also contains this u, since $U \subset U + W$ by definition of addition of subspaces. Any $u \in U$ is also in $U + W$, since we can express it as $u + 0w \in U + W$.
   But then, there exists no way of "removing" non-zero vectors in U from $U + W$, no matter what W we chose, since any $u \neq 0 \in U$ is also in $U + W$ by default.
   Thus the only subspace U for which we can have an additive inverse must actually be $U = \{0\}$. Such an inverse exists, since $\{0\} + \{0\} = \{0\}$.
   For an example as requested in the problem, consider $V = \mathbb{R}^2$. If we take

$$U_1 = \{(x,0), x \in \mathbb{R}\}, U_2 = \{(z,z), z \in \mathbb{R}\}, W = \{(0,y), y \in \mathbb{R}\}$$

   then both $U_1 + W$ and $U_2 + W$ will give you all of $\mathbb{R}^2$, yet $U_1 \neq U_2$. (HP)

Problem 3.3. Consider the example from class of *clock arithmetic* or *modular arithmetic* with modulus n: two integers a and b are called "equivalent" if a differs from b by a multiple of n. In this case, we write

$$a \equiv b \pmod{n}.$$

1. Show that n is composite (i.e., not prime) if and only if there are clock positions

$$a \not\equiv 0, \quad b \not\equiv 0 \pmod{n}$$

   such that $a \cdot b \equiv 0 \pmod{n}$.

2. Now let n be prime and consider an integer a with $a \not\equiv 0 \pmod{n}$.

   (a) Show that the function

$$f\colon \left\{ \begin{array}{c} \text{clock faces} \\ \text{with } n \text{ positions} \end{array} \right\} \to \left\{ \begin{array}{c} \text{clock faces} \\ \text{with } n \text{ positions} \end{array} \right\}$$

   defined by $f(b) \equiv a \cdot b \pmod{n}$ is *injective*.

(b) Use the finiteness of clock faces to show that f is also *surjective*.

(c) Conclude that there is a clock face b with $a \cdot b \equiv 1 \pmod{n}$.

3. Suppose that $n = 4 = 2 \cdot 2$, an example of a composite value of n. Set $a = 2$ and compute all the values of $f(b)$.

Solution. 1. Since we have an if and only if statement, let's prove both directions.
First, suppose n is composite. Then we want to find a and b as given in the problem. But by the definition of n being composite, there exist factors $a < n$ and $b < n$ that are not 1 or n themselves such that $a \cdot b = n$. Then these a and b are what we want - we have $a \not\equiv 0, b \not\equiv \pmod{n}$ and $a \cdot b = n = 0 (\text{mod } n)$.

To prove the other direction, consider the contrapositive. Suppose n is prime. Then we want to show that there exists no a and b that satisfy the conditions given in the problem.
Suppose that such a and b exist and satisfy the proper conditions. Then we have that $ab \equiv 0 (\text{mod } n)$, so that n, which is prime, must divide ab. But then by Euclid's first theorem, since n is prime, we then know that either n divides a or n divides b - in either case, one of a or b would be equivalent to $0 (\text{mod } n)$, a contradiction, showing us that if n is prime, such a and b do not exist.

2. (a) To show injectivity, suppose that $f(b_1) \equiv f(b_2)(\text{mod n})$, for $b_1, b_2 \in \{0, ..., n-1(\text{mod n})\}$. We want to show that $b_1 \equiv b_2(\text{mod n})$.
We then have that

$$f(b_1)(\text{mod n}) = ab_1(\text{mod n}) \equiv ab_2(\text{mod n}) = f(b_2)(\text{mod n})$$

Then ab_1 and ab_2 only differ by a number of full clock loops of size n, call it K, giving us

$$ab_1 = ab_2 + nK$$

for some K. Then we get

$$a(b_1 - b_2) = nK$$

But taking the remainder mod n on both sides gives us

$$a(b_1 - b_2) \text{ (mod n)} \equiv 0 \text{ (mod n)}$$

Since $a \not\equiv 0$ (mod n), we must have that

$$b_1 - b_2 \text{ (mod n)} \equiv 0$$

but then

$$b_1 \equiv b_2 \text{ (mod n)}$$

as we hoped to show.

(b) To show surjectivity, we first consider that we have n possible inputs (n clock positions) and n possible outputs (n clock positions). Yet, we just showed, via injectiveness above, that each distinct input must be mapped to a distinct output (in other words, no output positions can be repeated). Since we have each of the n input positions being mapped to a different position out of the n output positions, each of the output positions must be mapped to at least once, proving surjectivity.

(c) By surjectivity, which we just proved above, $f^{-1}(1 \text{ (mod n)}) \neq \emptyset$, so there must exist some b in the domain of f such that $f(b) \equiv a \cdot b \equiv 1$ (mod n).

3. If we set $a = 2$ and $n = 4$, then
$$f(b) \equiv 2 \cdot b \pmod 4$$
Then we get
$$\begin{aligned} f(0) &\equiv (2 \cdot 0) \pmod 4 = 0 \pmod 4 \\ f(1) &\equiv (2 \cdot 1) \pmod 4 = 2 \pmod 4 \\ f(2) &\equiv (2 \cdot 2) \pmod 4 = 4 \pmod 4 \equiv 0 \pmod 4 \\ f(3) &\equiv (2 \cdot 3) \pmod 4 = 6 \pmod 4 \equiv 2 \pmod 4 \end{aligned}$$
so that the only possible values of $f(b)$ are 0 and 2. (HP)

4 For submission to Rohil Prasad

Problem 4.1. Explain why there does *not* exist $\lambda \in \mathbb{C}$ such that
$$\lambda \cdot \begin{pmatrix} 2-3i \\ 5+4i \\ -6+7i \end{pmatrix} = \begin{pmatrix} 12-5i \\ 7+22i \\ -32-9i \end{pmatrix}.$$

Solution. Set $\lambda = a + bi$ for $a, b \in \mathbb{R}$.

We have
$$\begin{aligned} (a+bi)(2-3i) &= (2a+3b) + (2b-3a)i = 12-5i \\ (a+bi)(5+4i) &= (5a-4b) + (4a+5b)i = 7+22i \\ (a+bi)(-6+7i) &= (-6a-7b) + (7a-6b)i = -32-9i \end{aligned}$$

Looking at the real parts, we have $2a + 3b = 12$, $5a - 4b = 7$, and $-6a - 7b = -32$. Multiplying the first equation by 5 and the second by 2 and then subtracting, it follows that $23b = 60 - 14 = 46$, so $b = 2$. Plugging that into the third equation, we find $a = 3$.

However, by the imaginary part of the third equation above, we also have $7a - 6b = -9$. The LHS evaluates to $7 * 3 - 6 * 2 = 9$, so there can be no solution for λ. (RP)

Problem 4.2. Suppose that U_1 and U_2 are subspaces of a vector space V. Show that $U_1 \cup U_2$ is also a subspace if and only if one of the two subspaces is contained in the other.

Solution. We will prove the contrapositive and show that if neither of the subspaces is contained inside the other, then $U_1 \cup U_2$ cannot be a subspace.

Pick vectors u_1, u_2 such that u_1 is in U_1 but not U_2, and u_2 is in U_2 but not U_1.

By definition, both lie in $U_1 \cup U_2$. However, their sum $u_1 + u_2$ cannot lie in $U_1 \cup U_2$.

By definition, if it lies in $U_1 \cup U_2$ then it lies in U_1 or U_2. Assume that it lies in U_1. Then, since $-u_1 \in U_1$, we have $u_1 + u_2 - u_1 = u_2 \in U_1$. Since we originally assumed that $u_2 \notin U_1$, this is false. The same holds for the case where we assume that it lies in U_2.

Therefore, the sum does not lie in $U_1 \cup U_2$, so $U_1 \cup U_2$ is not closed under addition and therefore not a subspace. (RP)

Problem 4.3. In the definition of a vector space [Axler 1.19], there is the following condition:

> For each vector v there is an additive inverse vector $-v$ satisfying $v + (-v) = 0$.

Show that this condition can be replaced by the condition
$$0 \cdot v = 0.$$
(That is, show that either equation follows from the other using the other axioms of a vector space.)

Solution. Assume that every vector has an additive inverse.

We have $0 \cdot v = (0+0) \cdot v = 0 \cdot v + 0 \cdot v$. Subtracting the inverse of $0 \cdot v$ from both sides, we get $0 \cdot v = 0$.

Now assume $0 \cdot v = 0$. We have $(1 + (-1)) \cdot v = 0 \cdot v = 0$. The left-hand side distributes over addition and tells us that $v + (-1) \cdot v = 0$. Therefore, for any v we have $-1 \cdot v$ is its additive inverse. (RP)

Problem 4.4. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *periodic (with period $p \neq 0$)* if it satisfies

$$f(x+p) = f(x).$$

1. Fix a $p \neq 0$. Does the set of functions which are periodic with period p form a subspace of the vector space of all functions?

2. Now consider the set of *all* periodic functions, with unspecified period. (This is the union over all possible values of p of the sets considered in the first part.) Does this form a subspace of the vector space of all functions?

Solution. 1. It suffices to show that these functions are closed under addition and scalar multiplication.

Addition: For any $x \in \mathbb{R}$ and functions f, g we have

$$(f+g)(x+p) = f(x+p) + g(x+p) = f(x) + g(x) = (f+g)(x)$$

so the function $f + g$ is periodic with period p as desired.

Scalar Multiplication: For any $x \in \mathbb{R}$, function f and scalar $\lambda \in \mathbb{R}$, we have

$$(\lambda \cdot f)(x+p) = \lambda \cdot f(x+p) = \lambda \cdot f(x) = (\lambda \cdot f)(x)$$

so the function $\lambda \cdot f$ is periodic with period p as desired.

2. This does not form a subspace of $\mathbb{R}^{\mathbb{R}}$. We will show this by showing it is not closed under addition, i.e. by picking two periodic functions and showing that their sum is not periodic.

Let f be the function that takes a value 1 on every $x \in \mathbb{Z}$ and 0 elsewhere. Let g be the function that takes a value 1 on every $x = k\pi$ for $k \in \mathbb{Z}$ and 0 elsewhere. These functions have period 1 and π respectively.

Assume $f + g$ is periodic with period P. By the irrationality of π, there is no x that is both an integer and an integer multiple of π except for 0. Therefore, $f + g$ is 1 on integers or integer multiples of π, 2 at 0, and 0 elsewhere.

Therefore, $f + g$ cannot have period P since that would imply $f(P) = f(0) = 2$. Since $f + g$ is only equal to 2 at 0, we arrive at a contradiction. (RP)

Solution. Here's an alternative (but similar) counterexample to part 2, communicated to me by Emily Jia. Consider the functions

$$f(x) = \cos\left(\frac{2\pi x}{\sqrt{2}}\right), \qquad g(x) = \cos(2\pi x),$$

which have periods $\sqrt{2}$ and 1 respectively. We claim that the function $h(x) = f(x) + g(x)$ is not periodic for *any* period, which we will show by contradiction. Suppose that h were periodic for some period $p \geq 0$. At $x = 0$, h has a known value:

$$h(0) = f(0) + g(0) = \cos(0) + \cos(0) = 2.$$

By periodicity of h, it must also be the case that

$$h(p) = h(0+p) = h(0) = 2.$$

However, we also have a formula

$$h(p) = \cos\left(\frac{2\pi p}{\sqrt{2}}\right) + \cos(2\pi p).$$

Since this writes h as the sum of two functions with upper bound 1, it must be the case that f and g are both maximized at p — otherwise there is no way for their sum to give the value 2. However, we know exactly where the maxima of cosine occur: they are at the integer multiples of 2π. It follows that

$$\frac{2\pi p}{\sqrt{2}} = 2\pi k, \qquad\qquad 2\pi p = 2\pi \ell$$

for some integers k and ℓ. Combining these equations by eliminating p, we find

$$\frac{2\pi \ell}{\sqrt{2}} = 2\pi k,$$

or

$$\frac{\ell}{k} = \sqrt{2}.$$

Since $\sqrt{2}$ is irrational, this is not possible, and hence there cannot exist a period p. (ECP)