Homework #11

Math 25a

Due: November 30th, 2016

Guidelines:

- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one (though you should), a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!)

1 For submission to Thayer Anderson

Problem 1.1. Prove there does not exist an operator $f\colon \mathbb{R}^7 \to \mathbb{R}^7$ such that $f^2 + f + 1$ is nilpotent.

Problem 1.2. Suppose V is a real vector space and $f\colon V \to V$ is an operator. Suppose there exist $b, c \in \mathbb{R}$ such that $f^2 + bf + c = 0$. Prove that f has an eigenvalue if and only if $b^2 \geq 4c$.

Problem 1.3. Suppose V is a finite-dimensional real vector space and $f\colon V \to V$ is a linear operator. Show that the following are equivalent:

1. All the eigenvalues of $f_{\mathbb{C}}$ are real.
2. There exists a basis of V with respect to which f has an upper-triangular matrix.
3. There exists a basis of V consisting of generalized eigenvectors of f.

Problem 1.4. Throughout, let V denote a real vector space with an operator $f\colon V \to V$.

1. Suppose f has no eigenvalues. Conclude that $\det f > 0$.
2. Suppose $\dim V$ is even and that $\det f < 0$. Show f has at least two distinct eigenvalues.

2 For submission to Davis Lazowski

Problem 2.1. Give an example of an operator on a finite-dimensional real inner product space that admits an invariant subspace whose orthogonal complement is not invariant.

Problem 2.2. Let $e_1, \ldots, e_n$ be an orthonormal basis for an inner product space V, and let $f\colon V \to V$ be any linear operator. Show that the sum

$$\|fe_1\|^2 + \cdots + \|fe_n\|^2$$

is independent of the choice of basis.

Problem 2.3. Let V be an inner product space, and consider the vector space $\mathcal{L}(V, V)$ of operators on V. Show that

$$\langle f, g \rangle = \operatorname{tr}(f \circ g^*)$$

defines an inner product on $\mathcal{L}(V, V)$.

Problem 2.4. Suppose V is a real vector space and that $J\colon V \to V$ is a real linear operator satisfying $J^2 = -1$. Define complex scalar multiplication on V as follows: for $a, b \in \mathbb{R}$, set

$$(a + bi)v = av + bJv.$$

1. Show that this complex scalar multiplication and V's usual addition makes V into a complex vector space.
2. Show that the dimension of V as a complex vector space is half the dimension of V as a real vector space.

3 For submission to Handong Park

Problem 3.1. Suppose V is a real vector space and $f\colon V \to V$ has no eigenvalues. Show that every invariant subspace has even dimension.

Problem 3.2. Suppose that V is a real inner product space and that $f\colon V \to V$ is self-adjoint.

1. Show that $V_{\mathbb{C}}$ is a complex inner product space with inner product

$$\langle u + iv, x + iy \rangle_{\mathbb{C}} = \langle u, x \rangle + \langle v, y \rangle + (\langle v, x \rangle - \langle u, y \rangle)i.$$

2. Show that $f_{\mathbb{C}}$ is a self-adjoint operator on the inner product space $V_{\mathbb{C}}$.
3. Use complexification (and the previous two parts) to conclude the real spectral theorem (for self-adjoint real operators) from the complex spectral theorem (for normal complex operators).

Problem 3.3. Suppose V is an inner product space and that $f\colon V \to V$ is an operator satisfying the *hyponormality condition*

$$\|f^* v\| \leq \|fv\|.$$

Show that if V is finite-dimensional then f is automatically normal (i.e., the "inequality" is actually always an equality).

4 For submission to Rohil Prasad

Problem 4.1. Let $V_n = \operatorname{span}\{1, \cos x, \dots, \cos nx, \sin x, \dots, \sin nx\}$ be the vector space of functions considered over the previous few assignments, and let $D\colon V_n \to V_n$ be the differentiation operator on V_n. Having previously concluded that D was normal, find a basis for V_n such that the matrix presentation of D has the form guaranteed by Axler 9.34.

Problem 4.2. Suppose V is an inner product space and that $f\colon V \to V$ is an operator. Prove first that $\det f^* = \overline{\det f}$, then conclude

$$\det \sqrt{f^* f} = |\det f|.$$

Problem 4.3.

1. Give an example of a real vector space V and an operator $f\colon V \to V$ such that $\operatorname{tr}(f^2) < 0$.
2. Suppose that V is a real vector space and that $f\colon V \to V$ is an operator admitting a basis of eigenvectors. Show that $\operatorname{tr}(f^2) \geq 0$.

Problem 4.4. Suppose V is a real vector space with $\dim V = n$ and $f\colon V \to V$ is such that $\ker f^{\circ(n-1)} \neq \ker f^{\circ(n-2)}$. Prove that f has at most two distinct eigenvalues and that $f_{\mathbb{C}}$ has no nonreal eigenvalues.