Homework #11 Solutions

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1 For submission to Thayer Anderson

Problem 1.1. Prove there does not exist an operator $f\colon \mathbb{R}^7 \to \mathbb{R}^7$ such that $f^2 + f + 1$ is nilpotent.

Solution. Such an operator would have its characteristic polynomial dividing $(f^2+f+1)^N$ for some $N \gg 0$. However, $(f^2 + f + 1) = (f - \alpha)(f - \overline{\alpha})$ has only two complex-conjugate roots, which must appear in equal powers in the characteristic polynomial. At the same time, the characteristic polynomial must have total degree equal to the dimension of the space: 7. These contradict. (ECP)

Problem 1.2. Suppose V is a real vector space and $f\colon V \to V$ is an operator. Suppose there exist $b, c \in \mathbb{R}$ such that $f^2 + bf + c = 0$. Prove that f has an eigenvalue if and only if $b^2 \geq 4c$.

Solution. This proof is similar. Such an operator has minimal polynomial dividing $(f^2+bf+c)^N$ for $N \gg 0$, but since $(f^2 + bf + c)$ has two complex roots, the minimal polynomial must also have at most those two roots. If $b^2 \geq 4c$, then these two roots are real, and at least one of them induces (via the minimal polynomial) an eigenvalue of f. On the other hand, if f has no eigenvalues, then its minimal polynomial can have no real roots, forcing $b^2 < 4c$. (ECP)

Problem 1.3. Suppose V is a finite-dimensional real vector space and $f\colon V \to V$ is a linear operator. Show that the following are equivalent:

1. All the eigenvalues of $f_{\mathbb{C}}$ are real.
2. There exists a basis of V with respect to which f has an upper-triangular matrix.
3. There exists a basis of V consisting of generalized eigenvectors of f.

Solution. We address this in parts.

2. $\implies$ 1. The eigenvalues of $f_{\mathbb{C}}$ are read off of the main diagonal of any upper-triangular matrix presentation of it. Our assumption is that f admits an upper-triangular real presentation, which complexifies to an upper-triangular presentation of $f_{\mathbb{C}}$ with all real entries. In particular, its main diagonal consists of real numbers.

3. $\implies$ 2. If V has a basis of generalized eigenvectors, then V is exhausted by its generalized eigenspaces:

$$V = \bigoplus_{\lambda \in \mathbb{R}} G(\lambda, f).$$

From here, the same proof as in the complex case works: form a new basis of V by forming bases for $\ker(f-\lambda)^{\circ 1}$, extending that to bases for $\ker(f-\lambda)^{\circ 2}$, ..., on up to bases for $\ker(f-\lambda)^{\circ n}$.

1. $\implies$ 3. Start by forming a generalized eigenbasis for $f_{\mathbb{C}}$: $(u_j + iv_j)_{j=1}^n$. The collection $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ forms a spanning set for V, and it can thus be reduced to a basis for V. We claim that each of these

vectors is a generalized eigenvector: by assumption we have $(f_{\mathbb{C}} - \lambda_j)^{\circ n}(u_j + iv_j) = 0$, which expands via the definition of complexification to

$$(f - \lambda_j)^{\circ n}(u_j) + i(f - \lambda_j)^{\circ n}(v_j) = 0 + i0,$$

and examining this equation component-wise yields the desired result. (ECP)

Problem 1.4. Throughout, let V denote a real vector space with an operator $f\colon V \to V$.

1. Suppose f has no eigenvalues. Conclude that $\det f > 0$.
2. Suppose $\dim V$ is even and that $\det f < 0$. Show f has at least two distinct eigenvalues.

Solution. 1. The determinant of f agrees with the determinant of its complexification $f_{\mathbb{C}}$. The determinant of the complexification is the product of its eigenvalues (repeated according to algebraic multiplicity), which must all be non-real complex numbers occuring in conjugate pairs. Enumerating these as $\lambda_1, \overline{\lambda_1}, \lambda_2, \overline{\lambda_2}, \ldots, \lambda_k, \overline{\lambda_k}$, we thus have

$$\det f = \det f_{\mathbb{C}} = \prod_j \lambda_j \cdot \overline{\lambda_j} = \prod_j |\lambda_j|^2.$$

Finally, the product of positive numbers is positive.

2. Again, any complex eigenvalues of the complexification of f must come in conjugate pairs, the product of which gives a positive number. If $\det f < 0$, there must be some negative real number in the product. If the entire product were accounted for by conjugate pairs and a lone negative number, then the quantity of numbers (and hence the degree of the characteristic polynomial) would be odd, so there must be other real numbers also contributing to the product. (ECP)

2 For submission to Davis Lazowski

Problem 2.1. Give an example of an operator on a finite-dimensional real inner product space that admits an invariant subspace whose orthogonal complement is not invariant.

Solution. Any non-orthogonal projection will work.
Let $V = \mathbb{R}^2$, and let $p : V \to V$ be the projection

$$p(x, y) = (0, x + y)$$

Then $\operatorname{im} \mathrm{p} = \langle (0, 1) \rangle$, and $\langle (1, 0) \rangle = \langle (0, 1) \rangle^{\perp}$, but $p(1, 0) = (0, 1)$. (DL)

Problem 2.2. Let $e_1, \ldots, e_n$ be an orthonormal basis for an inner product space V, and let $f\colon V \to V$ be any linear operator. Show that the sum

$$\|fe_1\|^2 + \cdots + \|fe_n\|^2$$

is independent of the choice of basis.

Solution. We can expand the sum as:

$$\begin{aligned}\sum_{j=1}^{n} \langle fe_j, fe_j \rangle &= \sum_{j=1}^{n} \langle f^* fe_j, e_j \rangle \\ &= \sum_{j=1}^{n} \langle f^* fe_j, e_j \rangle\end{aligned}$$

f^*f is positive, so diagonalisable, so has an orthornormal diagonal basis $\xi_1 ... \xi_n$. We can expand the e_j in terms of it:

$$e_j = \sum_{i=1}^{n} \alpha_{ij} \xi_i$$

Therefore

$$\begin{aligned}
&\sum_{j=1}^{n} \langle f^* f e_j, e_j \rangle \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \langle f^* f \alpha_{ij} \xi_i, e_j \rangle \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_i \alpha_{ij} \langle \xi_i, e_j \rangle \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_i |\alpha_{ij}|^2 \langle \xi_i, \xi_i \rangle \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_i |\alpha_{ij}|^2
\end{aligned}$$

But because ξ_i is normal, $\sum_{j=1}^{n} |\alpha_{ij}|^2 = 1$. So this sum simplifies to

$$= \sum_{i=1}^{n} \lambda_i$$

Which is dependent only on f^*f and independent of basis. (DL)

Problem 2.3. Let V be an inner product space, and consider the vector space $\mathcal{L}(V, V)$ of operators on V. Show that

$$\langle f, g \rangle = \mathrm{tr}(f \circ g^*)$$

defines an inner product on $\mathcal{L}(V, V)$.

Solution.

- *Positive definite*: $f \circ f^*$ is a positive operator, so has positive eigenvalues and can be diagonalised, so $\mathrm{tr(f \circ f^*)} \geq 0$. If and only if all the eigenvalues are zero is their sum zero, in which case $f = 0$, as required.

- *Linear in first term*:

$$\begin{aligned}
&\langle f + \lambda h, g \rangle \\
&= \mathrm{tr([f + \lambda h] \circ g^*)} \\
&= \mathrm{tr(f \circ g^* + \lambda h \circ g^*)} \\
&= \mathrm{tr(f \circ g^*) + \lambda tr(h \circ g^*)}
\end{aligned}$$

  Where tr is linear because $[A + \lambda B]_{i,j} = [A]_{i,j} + \lambda [B]_{i,j}$, and tr is the sum of matrix diagonal elements.

- *Conjugation swaps elements*: Clearly, $\mathrm{tr(A^T)} = \mathrm{tr(A)}$ because transposition leaves the diagonal invariant. Also, $\mathrm{tr}(\overline{\mathrm{A}}) = \overline{\mathrm{tr(A)}}$ because conjugation is applied to each matrix diagonal element, and $\overline{a + b} = \overline{a} + \overline{b}$.Since $(f \circ g^*)^T = (\overline{g} \circ f^T) = \overline{g \circ f^*}$, then $\mathrm{tr(f \circ g^*)} = \overline{\mathrm{tr(g \circ f^*)}}$, so done. (DL)

Problem 2.4. Suppose V is a real vector space and that $J \colon V \to V$ is a real linear operator satisfying $J^2 = -1$. Define complex scalar multiplication on V as follows: for $a, b \in \mathbb{R}$, set

$$(a + bi)v = av + bJv.$$

1. Show that this complex scalar multiplication and V's usual addition makes V into a complex vector space.

2. Show that the dimension of V as a complex vector space is half the dimension of V as a real vector space.

Solution. 1. • *Additive axioms*: inherited from the underlying real vector space.

- *Multiplicative axioms*: Clearly $0x = 0$ and $1x = x$; these are inherited from the underlying real vector space. Also, $(ab)v = a(bv)$ because

$$
\begin{aligned}
[(a+bi)(c+di)]v &= [(ac-bd)+(ad+bc)i]v = (ac-bd)v + (ad+bc)Jv \\
(a+bi)[(c+di)v] &= (a+bi)[cv+dJv] = a(cv+dJv) + bJ(cv+dJv) \\
&= (ac)v + (ad)Jv + (bc)Jv - (bd)v = (ac-bd)v + (ad+bc)Jv
\end{aligned}
$$

  Where we use associativity and commuativity in the underlying real vector space freely.

- *Distributive axions*:

$$
\begin{aligned}
(a+bi)(v+u) &= a(v+u) + bJ(v+u) \\
= (a+bJ)v + (a+bJ)u &= (a+bi)v + (a+bi)u \\
[(a+bi)+(c+di)]v &= [(a+c)+(b+d)i]v \\
= (a+c)v + (b+d)Jv = (a+bJ)v + (c+dJ)v &= (a+bi)v + (c+di)v
\end{aligned}
$$

  As required.

2. The set $\{v, Jv\}$ is linearly independent in the underlying real vector space. If $Jv = \lambda v$, then $\lambda^2 = -1$, which is a contradiction.

   Construct a basis of the real vector space as follows: choose $v_0 \in V$. Then $\langle\{v_0, Jv_0\}\rangle = U_0$ as a two-dimensional subspace invariant under J, because $J(\alpha v_0 + \beta J v_0) = \alpha J v_0 - \beta v_0$. Choose $v_1 \notin U_0$. Let $U_1 = U_0 + \langle\{v_1, Jv_1\}\rangle$ to construct a four-dimensional subspace, and repeat. In this way, construct a basis $v_0, Jv_0, v_1, Jv_1 .... v_{n/2-1}, Jv_{n/2-1}$.

   Since this list spans the real vector space, it also spans the complex vector space. But $iv_0 = Jv_0$, so the list $v_0, v_1 ... v_{n/2-1}$ also spans the complex vector space.

   Also, all these vectors are linearly independent.

   Suppose
$$(\alpha_0 + i\beta_0)v_0 + ... + (\alpha_{n/2-1} + i\beta_{n/2-1})v_{n/2-1} = 0$$
   Then in the real vector space
$$\alpha_0 v_0 + ... + \alpha_{n/2-1} v_{n/2-1} + \beta_0 J v_0 + ... + \beta_{n/2-1} J v_{n/2-1} = 0$$
   Since the set $(v_0, Jv_0 ...)$ is linearly independent in the real vector space, then $\alpha_j = \beta_j = 0$ for all j. Therefore $v_0 ... v_{n/2-1}$ is linearly independent and spanning in the complex vector space, so is a basis, so the complex vector space has dimension $n/2$. (DL)

3 For submission to Handong Park

Problem 3.1. Suppose V is a real vector space and $f\colon V \to V$ has no eigenvalues. Show that every invariant subspace has even dimension.

Solution. If f has an invariant subspace U of odd dimension, we could restrict attention to $f|_U$, which is then a real operator on an odd-dimensional space and hence has an eigenvalue. A witnessing eigenvector v is also an eigenvector of f of the same weight. (ECP)

Problem 3.2. Suppose that V is a real inner product space and that $f\colon V \to V$ is self-adjoint.

1. Show that $V_{\mathbb{C}}$ is a complex inner product space with inner product
$$\langle u+iv, x+iy\rangle_{\mathbb{C}} = \langle u,x\rangle + \langle v,y\rangle + (\langle v,x\rangle - \langle u,y\rangle)i.$$
2. Show that $f_{\mathbb{C}}$ is a self-adjoint operator on the inner product space $V_{\mathbb{C}}$.
3. Use complexification (and the previous two parts) to conclude the real spectral theorem (for self-adjoint real operators) from the complex spectral theorem (for normal complex operators).

Solution. 1. We need to check the axioms: left linearity, right quasi-linearity, conjugate symmetry, and positive-definiteness.

Linearity:
$$\begin{aligned}\langle u+iv+u'+iv', x+iy\rangle_{\mathbb{C}} &= \langle (u+u')+i(v+v'), x+iy\rangle_{\mathbb{C}}\\ &= \langle u+u',x\rangle + \langle v+v',y\rangle + (\langle v+v',x\rangle - \langle u+u',y\rangle)i\\ &= \langle u,x\rangle + \langle v,y\rangle + (\langle v,x\rangle - \langle u,y\rangle)i + \langle u',x\rangle + \langle v',y\rangle + (\langle v',x\rangle - \langle u',y\rangle)i\\ &= \langle u+iv, x+iy\rangle_{\mathbb{C}} + \langle u'+iv', x+iy\rangle_{\mathbb{C}},\end{aligned}$$

and

$$\begin{aligned}\langle (a+bi)(u+iv), x+iy\rangle_{\mathbb{C}} &= \langle (au-bv)+i(av+bu), x+iy\rangle_{\mathbb{C}}\\ &= \langle au-bv,x\rangle + \langle av+bu,y\rangle + (\langle av+bu,x\rangle - \langle au-bv,y\rangle)i\\ &= a\langle u,x\rangle - b\langle v,x\rangle + a\langle v,y\rangle + b\langle u,y\rangle + (a\langle v,x\rangle + b\langle u,x\rangle - a\langle u,y\rangle + b\langle v,y\rangle)i\\ &= (a+bi)(\langle u,x\rangle + \langle v,y\rangle + (\langle v,x\rangle - \langle u,y\rangle)i)\\ &= (a+bi)\langle u+iv, x+iy\rangle_{\mathbb{C}}.\end{aligned}$$

Quasi-linearity:
$$\begin{aligned}\langle u+iv, x+iy+x'+iy'\rangle_{\mathbb{C}} &= \langle u+iv, (x+x')+i(y+y')\rangle_{\mathbb{C}}\\ &= \langle u,x+x'\rangle + \langle v,y+y'\rangle + (\langle v,x+x'\rangle - \langle u,y+y'\rangle)i\\ &= \langle u,x\rangle + \langle v,y\rangle + (\langle v,x\rangle - \langle u,y\rangle)i + \langle u,x'\rangle + \langle v,y'\rangle + (\langle v,x'\rangle - \langle u,y'\rangle)i\\ &= \langle u+iv, x+iy\rangle_{\mathbb{C}} + \langle u+iv, x'+iy'\rangle_{\mathbb{C}},\end{aligned}$$

and

$$\begin{aligned}\langle u+iv, (a+ib)(x+iy)\rangle_{\mathbb{C}} &= \langle u+iv, (ax-by)+i(bx+ay)\rangle_{\mathbb{C}}\\ &= \langle u,ax-by\rangle + \langle v,bx+ay\rangle + (\langle v,ax-by\rangle - \langle u,bx+ay\rangle)i\\ &= a\langle u,x\rangle + b\langle v,x\rangle - b\langle u,y\rangle + a\langle v,y\rangle + (a\langle v,x\rangle - b\langle u,x\rangle - b\langle v,y\rangle - a\langle u,y\rangle)i\\ &= (a-bi)(\langle u,x\rangle + \langle v,y\rangle + (\langle v,x\rangle - \langle u,y\rangle)i)\\ &= \overline{(a+bi)}\langle u+iv, x+iy\rangle_{\mathbb{C}}.\end{aligned}$$

Symmetry:
$$\begin{aligned}\langle u+iv, x+iy\rangle_{\mathbb{C}} &= \langle u,x\rangle + \langle v,y\rangle + (\langle v,x\rangle - \langle u,y\rangle)i\\ &= \langle x,u\rangle + \langle y,v\rangle + (\langle y,u\rangle - \langle x,v\rangle)(-i)\\ &= \overline{\langle x+iy, u+iv\rangle_{\mathbb{C}}}.\end{aligned}$$

P.D.:

$$\begin{aligned}\langle u+iv, u+iv\rangle_{\mathbb{C}} &= \langle u,u\rangle + \langle v,v\rangle + (\langle v,u\rangle - \langle u,v\rangle)i \\ &= \|u\|^2 + \|v\|^2 + 0i.\end{aligned}$$

This quantity is always non-negative, and it is zero exactly when both $u = 0$ and $v = 0$.

2. We check that $f_{\mathbb{C}}$ has the inner-product property that $(f_{\mathbb{C}})^*$ demands:

$$\begin{aligned}\langle f_{\mathbb{C}}(u+iv), x+iy\rangle_{\mathbb{C}} &= \langle f(u)+if(v), x+iy\rangle_{\mathbb{C}} \\ &= \langle fu, x\rangle + \langle fv, y\rangle + (\langle fv, y\rangle - \langle fu, x\rangle)i \\ &= \langle u, fx\rangle + \langle v, fy\rangle + (\langle v, fy\rangle - \langle u, fx\rangle)i \\ &= \langle u+iv, fx+ify\rangle_{\mathbb{C}} \\ &= \langle u+iv, f_{\mathbb{C}}(x+iy)\rangle_{\mathbb{C}}.\end{aligned}$$

3. Beginning with a self-adjoint operator f on a real inner-product space V, we have shown that $f_{\mathbb{C}}$ is a self-adjoint operator on the complexified inner-product space $V_{\mathbb{C}}$. The complex spectral theorem thus says that $f_{\mathbb{C}}$ is diagonalizable on $V_{\mathbb{C}}$ in an orthonormal basis $(u_j + iv_j)_{j=1}^n$, and in that eigenbasis it presents as a real diagonal matrix. From here, we proceed as in Problem 1.3: the list $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ spans V and are eigenvectors for f, and Gram–Schmidt (modified to discard vectors that it nullifies) reduces this to an orthonormal list $(w_1, \ldots, w_n)$. Finally, this orthonormal set is also an eigenbasis: Gram–Schmidt modifies each vector by an orthogonal projection, and eigenspaces for normal operators are mutually perpendicular, so each vector remains within its eigenspace. (ECP)

Problem 3.3. Suppose V is an inner product space and that $f\colon V \to V$ is an operator satisfying the *hyponormality condition*

$$\|f^*v\| \le \|fv\|.$$

Show that if V is finite-dimensional then f is automatically normal (i.e., the "inequality" is actually always an equality).

Solution. This is a trace problem. Start by selecting an orthonormal basis (e_j) of V, and use the equality

$$\operatorname{tr}(f^*f) = \sum_j \|fe_j\|^2.$$

Because the trace is invariant under cyclic permutation, we also have

$$\operatorname{tr}(f^*f) = \operatorname{tr}(ff^*) = \operatorname{tr}((f^*)^*f^*) = \sum_j \|f^*e_j\|^2.$$

The inequality $\|fv\| \ge \|f^*v\|$ lets us compare these sums termwise: it must be the case that $\|fe_j\| = \|f^*e_j\|$ for each j. Finally, the Pythagorean theorem lets us calculate $\|fv\|$ for any $v = k_1e_1 + \cdots + k_ne_n$:

$$\begin{aligned}\|fv\|^2 &= \|f(k_1e_1 + \cdots + k_ne_n)\|^2 \\ &= |k_1|^2\|fe_1\|^2 + \cdots + |k_n|^2\|fe_n\|^2 \\ &= |k_1|^2\|f^*e_1\|^2 + \cdots + |k_n|^2\|f^*e_n\|^2 \\ &= \|f^*(k_1e_1 + \cdots + k_ne_n)\|^2 = \|f^*v\|^2.\end{aligned} \quad \text{(ECP)}$$

4 For submission to Rohil Prasad

Problem 4.1. Let $V_n = \operatorname{span}\{1, \cos x, \dots, \cos nx, \sin x, \dots, \sin nx\}$ be the vector space of functions considered over the previous few assignments, and let $D\colon V_n \to V_n$ be the differentiation operator on V_n. Having previously concluded that D was normal, find a basis for V_n such that the matrix presentation of D has the form guaranteed by Axler 9.34.

Solution. Rearrange the basis to take the form

$$\{1, \sin x, \cos x, \sin 2x, \cos 2x, \dots, \sin nx, \cos nx\}.$$

The matrix presentation is then

$$D = \begin{pmatrix} 0 & & & & & & & \\ & 0 & -1 & & & & & \\ & 1 & 0 & & & & & \\ & & & 0 & -2 & & & \\ & & & 2 & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & -n \\ & & & & & & n & 0 \end{pmatrix},$$

with zeroes in the omitted positions. (ECP)

Problem 4.2. Suppose V is an inner product space and that $f\colon V \to V$ is an operator. Prove first that $\det f^* = \overline{\det f}$, then conclude

$$\det \sqrt{f^* f} = |\det f|.$$

Solution. Determinants can be computed using a matrix formula, and an orthonormal matrix presentation of f gives rise to an orthonormal matrix presentation of f^* by taking its conjugate transpose. Rearranging the sum to range over rows rather than columns shows $\det f^* = \overline{\det f}$. Additionally, since $\sqrt{f}$ satisfies $\sqrt{f} \circ \sqrt{f} = f$, we have $\det \sqrt{f} \cdot \det \sqrt{f} = \det f$, hence $\det \sqrt{f} = \sqrt{\det f}$. Altogether, this gives

$$\det \sqrt{f^* f} = \sqrt{\det f^* f} = \sqrt{\det f^* \cdot \det f} = \sqrt{\overline{\det f} \cdot \det f} = \sqrt{|\det f|^2} = |\det f|. \qquad \text{(ECP)}$$

Problem 4.3.

1. Give an example of a real vector space V and an operator $f\colon V \to V$ such that $\operatorname{tr}(f^2) < 0$.
2. Suppose that V is a real vector space and that $f\colon V \to V$ is an operator admitting a basis of eigenvectors. Show that $\operatorname{tr}(f^2) \geq 0$.

Solution.

1. Take $f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to be the rotation operator, with square $f^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ the double reflection operator. This has $\operatorname{tr} f^2 = -2$.
2. The eigenbasis of f diagonalizes it. The square of a diagonal matrix is computed entry-wise, which means that all the entries of the diagonal presentation of f^2 are nonnegative. The trace is the sum of the diagonal elements, so it too is nonnegative. (ECP)

Problem 4.4. Suppose V is a real vector space with $\dim V = n$ and $f\colon V \to V$ is such that $\ker f^{\circ(n-1)} \neq \ker f^{\circ(n-2)}$. Prove that f has at most two distinct eigenvalues and that $f_{\mathbb{C}}$ has no nonreal eigenvalues.

Solution. The inequality $\ker f^{\circ(n-1)} \neq \ker f^{\circ(n-2)}$ states that the generalized eigenspace of weight 0 has dimension at least $n-1$. If f had two distinct other eigenvalues, then they would both have witnessing eigenvectors which would, in turn, give a contradicting bound $\dim G(0, f) \leq n-2$. If f had a nonreal eigenvalue, its complex conjugate would also be a nonreal eigenvalue, fueling the same contradiction. (ECP)